# Optimal space trajectories

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# Space trajectories

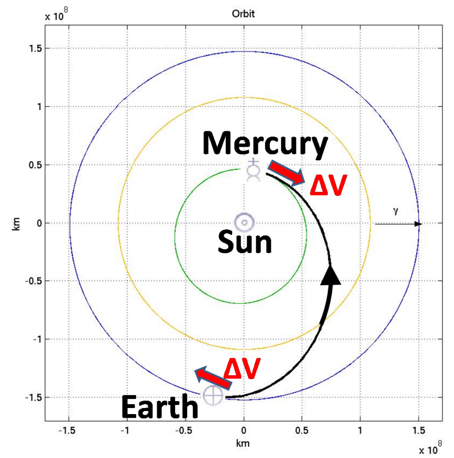
- European Space Agency
- Mission Analysis Office (ESOC, Darmstadt, Germany)
- Design and ***optimize*** trajectories for interplanetary space missions

# Space trajectories Example#1: Mission to Mercury I

- ESA cornerstone missions: BepiColombo
- Two spacecraft in orbit around Mercury (magnetospheric and planetary orbiter)
- Launch : 2014
- Cost: 1.5 billion $

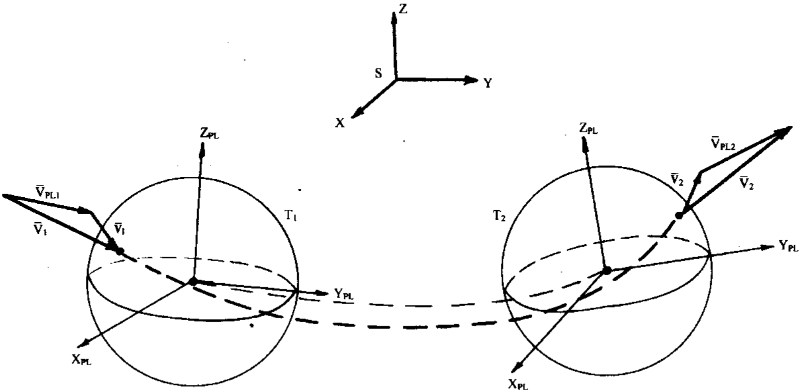
# Space trajectories Example#1: Mission to Mercury II

- Fast trajectory: 100 days
- Total $\Delta V = 16$ km/s
- $M_f = M_0 \exp(-\frac{\Delta V}{g I_{sp}})$
- $g = 9.8\, m/s^2$; $I_{sp} = 300\, s$ (chemical propulsion - minutes/hours)
- $M_f / M_0 < 1\%$



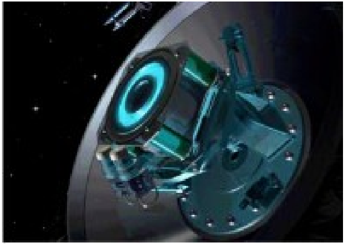
# Space trajectories Example#1: Mission to Mercury III

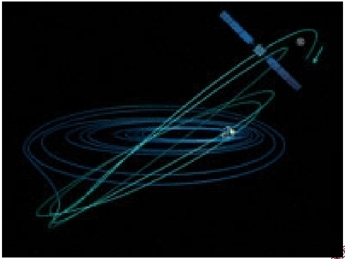
Gravity assists (reducing $\Delta V$)



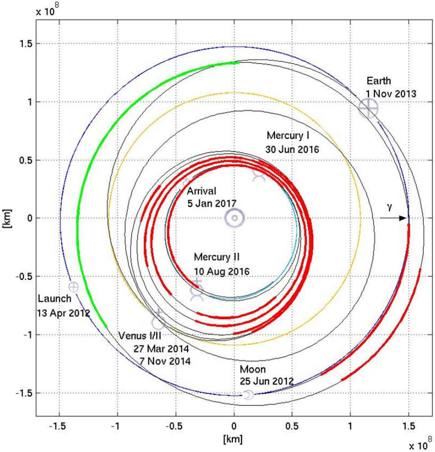
# Space trajectories Example#1: Mission to Mercury IV

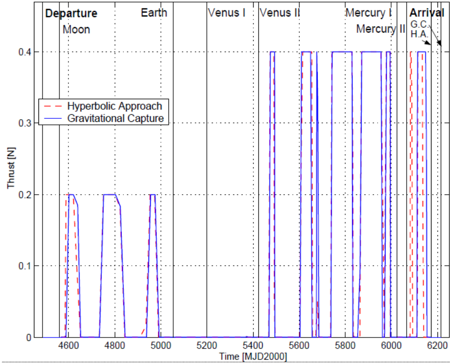
- Electric propulsion
- $I_{sp} = 4000\,s$, increase $M_f$
- Low-thrust $0.1\,N$ ...
- ...over long arcs (days/months)
- Power (solar array) and Xenon
- Example: Smart1 mission to the Moon (2003-2006)
- *Optimal thrust law*



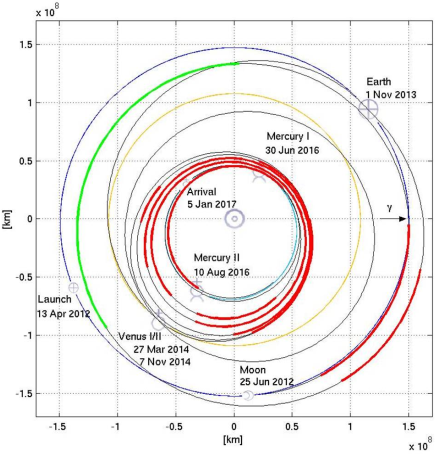


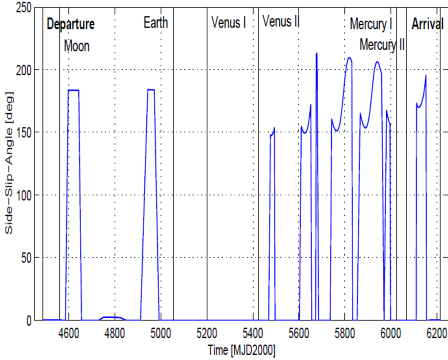
# Space trajectories Example#1: Mission to Mercury V



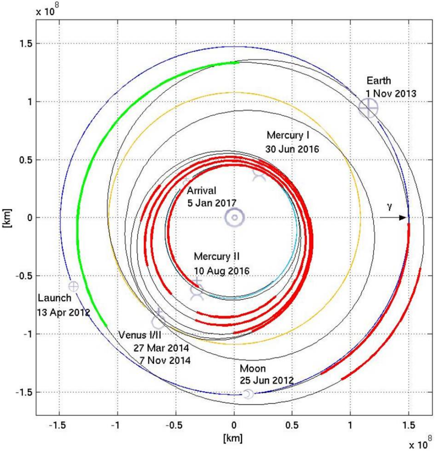



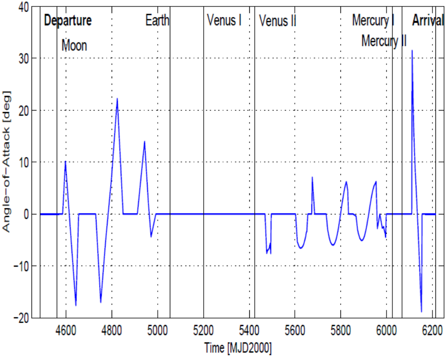
# Space trajectories Example#1: Mission to Mercury VI



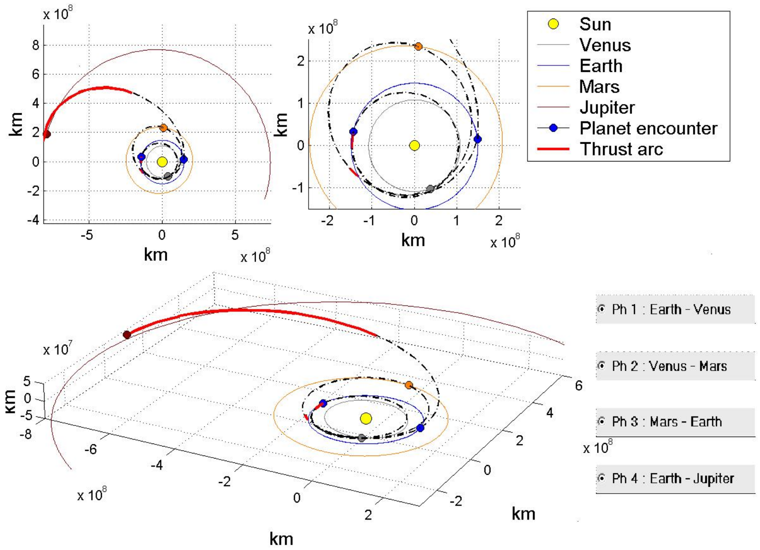



# Space trajectories Example#1: Mission to Mercury VII

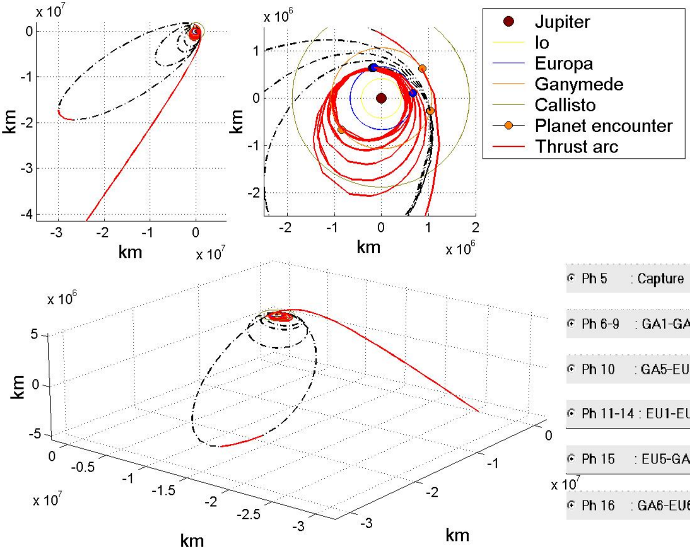


# Space trajectories Example #2: Mission to Europa



# Space trajectories Example #2: Mission to Europa




- Ph 5 : Capture
- Ph 6-9 : GA1-GA5 tour
- Ph 10 : GA5-EU1 transfer
- Ph 11-14 : EU1-EU5 tour
- Ph 15 : EU5-GA6 transfer
- Ph 16 : GA6-EU6 transfer

# Optimal control theory Brief overview I

## Definition

**Optimal control problem** : find $\overline{x}(t), \overline{u}(t), \overline{t}_0, \overline{t}_1$ that minimize the merit functional

$$J = \varphi(t_0, x(t_0), t_1, x(t_1)) + \int_{t_0}^{t_1} L(x(t), u(t), t)\, dt \tag{1}$$

subject to the dynamic constraints

$$\dot{x}(t) = f(x(t), u(t), t) \tag{2}$$

and to the boundary constraints

$$\psi(t_0, x(t_0), t_1, x(t_1)) = 0 \tag{3}$$

# Optimal control theory Brief overview II

where

$$t \in I = [t_0, t_1]$$

$$\varphi(t_0, x(t_0), t_1, x(t_1)),\; L(x(t), u(t), t) \in \mathbb{R}$$

$$x(t) \in \mathbb{R}^n,\, x(\cdot) \in \widehat{C}^1(I, \mathbb{R}^n)$$

$$u(t) \in U \subseteq \mathbb{R}^m,\, u(\cdot) \in \widehat{C}^0(I, U)$$

$$\psi(t_0, x(t_0), t_1, x(t_1)) \in \mathbb{R}^p,\, p \leq 2n+2$$

$$f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t}, L, \frac{\partial L}{\partial x}, \frac{\partial L}{\partial t} \text{ continuos in } (x, u, t)$$

**Remarks**

1. $U$ can be a proper, closed subset of $\mathbb{R}^m$ (constrained controls)
2. If the cost functional is as in Eq. (1), we have a *problem of Bolza*. Equivalent formulations are

   2.1 The problem of Lagrange $J = \int_{t_0}^{t_1} L(x(t), u(t), t)\, dt$

   2.2 The problem of Meyer $J = \varphi(t_0, x(t_0), t_1, x(t_1))$

   NOTE: We can transform a problem of Lagrange into a problem of Meyer with the additional variable y(t) and the constraint $\dot{y}(t) = L(x(t), u(t), t)$ and $y(t_0) = 0$
3. The dynamics constraints Eq.(2) can represent a second order dynamical systems. E.g.

   3.1 $x = \begin{pmatrix} r \\ v \end{pmatrix}, f(x, u, t) = \begin{pmatrix} v \\ F(r) + u \end{pmatrix}$

   3.2 $x = \begin{pmatrix} q \\ p \end{pmatrix}, f(x, u, t) = \begin{pmatrix} \frac{\partial \mathscr{H}}{\partial p} \\ -\frac{\partial \mathscr{H}}{\partial q} + u \end{pmatrix}$

# Optimal control theory Brief overview IV

4. The boundary constraints Eq.(3) include fixed initial/final times, fixed initial/final states, etc. E.g.

   4.1 $t_0 = 0, t_1 = 1, x(t_0) = x_0$ becomes $\psi = \begin{pmatrix} t_0 \\ t_1 - 1 \\ x(t_0) - x_0 \end{pmatrix} = 0$

5. Can be extended to manifolds: $x(t) \in M$

Indirect methods implement the necessary conditions given by the Pontryagin principle to find $\overline{x}(t), \overline{u}(t), \overline{t}_0, \overline{t}_1$.
How does the Pontryagin principle define the necessary conditions?

- Assume we have the optimal control $\overline{x}(t), \overline{u}(t), \overline{t}_0, \overline{t}_1$
- Apply the *arbitrary (but allowed!) variations* $\delta x, \delta u, \delta t_0, \delta t_1$, defined by e.g. $\frac{d}{d\varepsilon} x_\varepsilon(t)|_{\varepsilon=0}$
- Compute the *first variation* $\delta J$
- Impose $\delta J \geq 0$

It's tricky to compute $\delta J$ for variations that satisfy all the constraints Eq.(2-3). Then we use the **augmented problem**.

# Indirect methods Augmented problem I

- Introduce the costates $\lambda(\cdot) \in \widehat{C}^1\left(I, (\mathbb{R}^n)^*\right)$ and the multipliers $\nu \in (\mathbb{R}^p)^*$

*$\left(\mathbb{R}^k\right)^*$ is the dual space of $\mathbb{R}^k$. An element of $\left(\mathbb{R}^k\right)^*$ pairs with an element of $T\mathbb{R}^k$ to give a scalar: $\langle \lambda, \dot{x} \rangle \in \mathbb{R}$ or $\langle \nu, \psi \rangle \in \mathbb{R}$. We can identify $\mathbb{R}^k$ with $\left(\mathbb{R}^k\right)^*$ and $T\mathbb{R}^k$, and write the pairing in components as*

$$\langle \lambda, \dot{x} \rangle = \lambda_i \dot{x}^i \quad \langle \nu, \psi \rangle = \nu_i \psi^i$$

- Introduce the control Hamiltonian

$$H(x, \lambda, u, t) = L(x, u, t) + \langle \lambda, f(x, u, t) \rangle \tag{4}$$

- Introduce $\Phi^\nu = \varphi + \langle \nu, \psi \rangle$
- Introduce $J^\nu = \Phi^\nu + \int_{t0}^{t1} \left(H - \langle \lambda, \dot{x} \rangle\right) dt$

# Indirect methods Augmented problem II

## Definition

**Augmented optimal control problem:** minimize the merit functional $J^\nu$ subject to Eq.(2-3)

__Remarks__

1. If $\overline{x}(t), \overline{u}(t), \overline{t}_0, \overline{t}_1$ is a solution to optimal control problem, then it is also a solution of the augmented problem *for arbitrary values of $\lambda$ and $\nu$.*
2. Paradox: don't we have a more difficult problem to solve (both constrained optimization problems)? Yes, but we can choose $\lambda$ and $\nu$ , which are arbitrary, to simplify $\delta J$. For instance, we can choose them to delete the terms that multiply the variations $\delta x, \delta t_0, \delta t_1$, which should be otherwise computed as functions of $\delta u$.

# Indirect methods Pontryagin minimum principle I

By imposing $\delta J \geq 0$ we find the following necessary conditions

$$\overline{u}(\overline{x}(t), \lambda(t), t) = \arg \min_{u(t)\in U} H(\overline{x}(t), \lambda(t), u(t), t) \tag{5}$$

$$\dot{\overline{x}}(t) = f(\overline{x}(t), \lambda(t), t) = \left.\frac{\partial H(x, \lambda, u, t)}{\partial \lambda}\right|_{u=\overline{u}(\overline{x},\lambda,t),\, x=\overline{x}} \tag{6}$$

$$\dot{\overline{\lambda}}(t) = -\left.\frac{\partial H(x, \lambda, u, t)}{\partial x}\right|_{u=\overline{u}(\overline{x},\lambda,t),\, x=\overline{x}} \tag{7}$$

together with the b.c. Eq.(3) and the transversality conditions

$$\left(\frac{\partial \Phi^{\nu}}{\partial t_0} - H_{(0)}\right)\delta t_0 + \left(\frac{\partial \Phi^{\nu}}{\partial t_1} + H_{(1)}\right)\delta t_1 + \left(\frac{\partial \Phi^{\nu}}{\partial x(t_0)} + \lambda(t_0)\right)\delta(x(t_0)) + \left(\frac{\partial \Phi^{\nu}}{\partial x(t_1)} - \lambda(t_1)\right)\delta(x(t_1)) = 0$$

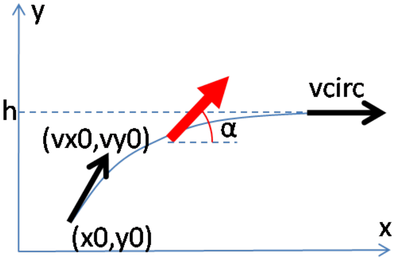
where here $H_{(i)} = H(x(t_i), \lambda(t_i), u(t_i), t_i)\,, i = 0, 1$

# Indirect methods Pontryagin minimum principle II

**Remarks**

1. If $U = \mathbb{R}^m$ (but $U$ open would suffice) then Eq.(5) is often replaced by
$$\frac{\partial H}{\partial u} = 0\,, \; \frac{\partial^2 H}{\partial u^2} \geq 0 \tag{8}$$
However, Eq.(5) is a stronger condition: the optimal control at any given time minimizes the Hamiltonian over the set of all possible controls. Eq. (8) only gives the necessary conditions for a *local* minimum of $H$ with respect to $u$.
2. Even if Eq.(5) says that at any time $t$, $u(t)$ is global optimum for $H$, the necessary conditions for optimality of $J$ as function of $u(\cdot), x(\cdot), t_0, t_1$ are only local!
3. Eq.(6-7) is a $2n$ dimensional dynamical system: we need $2n+2$ b.c. (e.g, initial and final time + $2n$ i.c.). The $2n+2$ b.c. and the $p$ parameters $\nu$ are found with the $p$ b.c. Eq.(3) and with the $2n+2$ equations from the transversality conditions.

# Indirect methods Example #1: linear tangent steering law I




- Launch into circular orbit from flat Earth.
- Very simplifying assumptions, yet it was used by the guidance system of the SaturnV (that sent the men on the Moon), with some refinements.
- we want to minimize the fuel mass: because we assume constant thrust, we equivalently minimize the final time.

# Indirect methods Example #1: linear tangent steering law II

- Problem formulated as follow: minimize

$$J = t_1$$

subject to the dynamic constraints

$$\begin{cases} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{v}_x &= \frac{F}{m_o + \dot{m}t}\cos\alpha \\ \dot{v}_y &= \frac{F}{m_o + \dot{m}t}\cos\alpha - g \end{cases}$$

and to the boundary constraints

$$\begin{cases} t_0 &= 0 \\ x(t_0) &= x_0 \\ y(t_0) &= y_0 \\ v_x(t_0) &= v_{x0} \\ v_y(t_0) &= v_{y0} \\ y(t_1) &= h \\ v_x(t_1) &= v_{circ} \\ v_y(t_1) &= 0 \end{cases}$$

# Indirect methods Example #1: linear tangent steering law III

1. Compute the Hamiltonian

$$H = \langle \lambda, f \rangle = \lambda^1 v_x + \lambda^2 v_y + \lambda^3 \frac{F}{m} \cos\alpha + \lambda^4 \left( \frac{F}{m} \sin\alpha - g \right)$$

2. Compute the costate equations

$$\begin{cases} \dot{\lambda}^1 & = -\frac{\partial H}{\partial x} = & 0 \\ \dot{\lambda}^2 & = -\frac{\partial H}{\partial y} = & 0 \\ \dot{\lambda}^3 & = -\frac{\partial H}{\partial v_x} = & -\lambda^1 \\ \dot{\lambda}^4 & = -\frac{\partial H}{\partial v_x} = & -\lambda^2 \end{cases} \rightarrow \begin{cases} \lambda^1 & = & c_1 \\ \lambda^2 & = & c_2 \\ \lambda^3 & = & -c_1 t + c_3 \\ \lambda^4 & = & -c_2 t + c_4 \end{cases}$$

# Indirect methods Example #1: linear tangent steering law IV

3. Compute the optimal control

$$\frac{\partial H}{\partial \alpha} = -\lambda^3 \frac{F}{m} \sin\alpha + \lambda^4 \frac{F}{m} \cos\alpha = 0 \rightarrow \tan\alpha = \frac{\lambda^4}{\lambda^3} = \frac{-\lambda^4}{-\lambda^3}$$

Thus

$$\cos\alpha = \frac{\pm\lambda^3}{\sqrt{(\lambda^3)^2 + (\lambda^4)^2}}$$

$$\sin\alpha = \frac{\pm\lambda^4}{\sqrt{(\lambda^3)^2 + (\lambda^4)^2}}$$

To solve the ambiguity of the sign, let's compute

$$\frac{\partial^2 H}{\partial \alpha^2} = -\frac{F}{m}\left(\pm\sqrt{(\lambda^3)^2 + (\lambda^4)^2}\right)$$

# Indirect methods Example #1: linear tangent steering law V

Because we want $\frac{\partial^2 H}{\partial \alpha^2} \geq 0$, we choose the minus sign

$$\cos\alpha = \frac{-\lambda_3}{\sqrt{(\lambda^3)^2 + (\lambda^4)^2}}$$

$$\sin\alpha = \frac{-\lambda_4}{\sqrt{(\lambda^3)^2 + (\lambda^4)^2}}$$

4. Compute the transversality conditions

$$\left(\frac{\partial \Phi^\nu}{\partial t_0} - H_{(0)}\right)\delta t_0 + \left(\frac{\partial \Phi^\nu}{\partial t_1} + H_{(1)}\right)\delta t_1 + \left(\frac{\partial \Phi^\nu}{\partial X(t_0)} + \lambda(t_0)\right)\delta\left(X(t_0)\right) + \left(\frac{\partial \Phi^\nu}{\partial X(t_1)} - \lambda(t_1$$

where $X = (x, y, v_x, v_y)$. Because the initial time and position are fixed, $\delta t_0 = \delta(X(t_0)) = 0$; also the final $y, v_x, v_y$ are fixed, so $\delta\left(y(t_1)\right) = \delta\left(v_x(t_1)\right) = \delta\left(v_y(t_1)\right) = 0$. The differential reduces to

$$\left(\frac{\partial \Phi^\nu}{\partial t_1} + H_{(1)}\right)\delta t_1 + \left(\frac{\partial \Phi^\nu}{\partial x(t_1)} - \lambda^1(t_1)\right)\delta\left(x(t_1)\right) = 0$$

# Indirect methods Example #1: linear tangent steering law VI

which results in the two equations

$$H_{(1)} = -\frac{\partial \Phi^{\nu}}{\partial t_1} = -1$$

$$\lambda^1(t_1) = \frac{\partial \Phi^{\nu}}{\partial x(t_1)} = 0$$

5. Summary: the optimal trajectory is solution of the 8th dimensional dynamical system

$$\begin{cases} \dot{x} & = & v_x \\ \dot{y} & = & v_y \\ \dot{v}_x & = & \frac{F}{m_o + \dot{m}t}\cos\alpha \\ \dot{v}_y & = & \frac{F}{m_o + \dot{m}t}\cos\alpha - g \\ \dot{\lambda}^1 & = & 0 \\ \dot{\lambda}^2 & = & 0 \\ \dot{\lambda}^3 & = & -\lambda^1 \\ \dot{\lambda}^4 & = & -\lambda^2 \end{cases}$$

# Indirect methods Example #1: linear tangent steering law VII

with the 10 boundary conditions

$$\begin{cases} t_0 & = & 0 \\ x(t_0) & = & x_0 \\ y(t_0) & = & y_0 \\ v_x(t_0) & = & v_{x0} \\ v_y(t_0) & = & v_{y0} \\ y(t_1) & = & h \\ v_x(t_1) & = & v_{circ} \\ v_y(t_1) & = & 0 \\ \lambda_1(t_1) & = & 0 \\ H_1 & = & -1 \end{cases}$$

Note that we can use the final conditions $\lambda_1(t_1) = 0$ to find $c_1 = 0$. Then the optimal control is the tangent linear control law

$$\tan\alpha = \frac{-\lambda^4}{-\lambda^3} = -\frac{c_2}{c_3}t + \frac{c_4}{c_3}$$

# Indirect methods Example #1: linear tangent steering law VIII

## IMPORTANT REMARKS

- In order to compute the optimal trajectory, we need to solve a two-point boundary-value problem (2pbvp), because some of the boundary conditions are at the initial time, some others are at a final time. This is the strategy:
  - Guess a value for the five variables $\lambda^1(t_0), \lambda^2(t_0), \lambda^3(t_0), \lambda^4(t_0), t_1$.
  - Integrate the system of 8 ODEs.
  - Verify that the final conditions match the 5 final constraints. If not, use the residual to correct the initial guesses and repeat.
- Solving the 2bpvp is the major problem of indirect methods:
  - Guessing the value of the costate is not easy - there is huge literature dedicated to this problem.
  - The differential corrector schemes required to solve the 2pbvp are not simple to implement, especially when the vector field is not continuous (see bang-bang control).
  - The system is highly nonlinear, i.e. very sensitive to the initial conditions.

# Indirect methods Primer vector theory I

1. We now consider dynamical systems like

$$\begin{cases} \dot{r} = v \\ \dot{v} = F(r) + G(v) + u \end{cases}$$

with the assumption

$$\frac{\partial F}{\partial r} = \left(\frac{\partial F}{\partial r}\right)^T, \frac{\partial G}{\partial v} = -\left(\frac{\partial G}{\partial v}\right)^T \tag{9}$$

2. We consider merit function $J = \varphi + \int L dt$ where the control Lagrangian can be
   2.1 $L = \frac{1}{2}\|u\|^2 \rightarrow J = \int \frac{1}{2}\|u\|^2 dt$ Minimum control effort
   2.2 $\frac{\partial L}{\partial u} = 0$ e.g:
      - $L = 1 \rightarrow J = \int 1 dt = t_1 - t_0$ Minimum transfer time
      - $L = 0 \rightarrow J = m(t_0) - m(t_1)$ Minimum fuel mass

# Indirect methods Primer vector theory II

- The primer vector theory shows that $u$ is aligned with $\delta r$ and with $\lambda_v$, where $\lambda_v$ is the costate associated to the velocity, and $\delta r$ is solution of the linearized system

$$\delta \ddot{r}^i = \frac{\partial F^i}{\partial r^j} \delta r^j + \frac{\delta G^i}{\delta \dot{r}^j} \delta \dot{r}^j \tag{10}$$

- The optimal control vector points towards a neighbor moving point being subject to the same vector field. (Marec)
- Nice interpretation of the costates

**Proof**

1. Compute the control Hamiltonian

$$
\begin{aligned}
H &= L + \langle \lambda_r, v \rangle + \langle \lambda_v, F(r) + G(v) + u \rangle \\
&= L + (\lambda_r)_i \, v^i + (\lambda_v)_i \left( F^i(r) + G^i(v) + u^i \right)
\end{aligned}
$$

2. Compute the costates equations

$$
\begin{cases}
\left(\dot{\lambda}_r\right)_i = -\frac{\partial H}{\partial r^i} = -(\lambda_v)_j \frac{\partial F^j}{\partial r^i} \\
\left(\dot{\lambda}_v\right)_i = -\frac{\partial H}{\partial v^i} = -(\lambda_r)_j - (\lambda_v)_j \frac{\partial G^j}{\partial r^i}
\end{cases}
\tag{11}
$$

## Indirect methods Primer vector theory IV

3. Differentiate the second Eq.(11) and use the first Eq.(11), and Eq.(9)

$$\left(\ddot{\lambda}_v\right)_i = -\left(\dot{\lambda}_r\right)_j - \left(\dot{\lambda}_v\right)_j \frac{\partial G^j}{\partial r^i} =$$
$$= (\lambda_v)_j \frac{\partial F^j}{\partial r^i} - \left(\dot{\lambda}_v\right)_j \frac{\partial G^j}{\partial r^i} =$$
$$= (\lambda_v)_j \frac{\partial F^i}{\partial r^j} + \left(\dot{\lambda}_v\right)_j \frac{\partial G^i}{\partial r^j}$$

   This shows that $\lambda_v$ is a solution of the linearized system Eq.(10) ($\lambda_v$ is the aligned with $\delta r$)

4. Minimize H to find the optimal control
   - 4.1 $H = \frac{1}{2}\|u\|^2 + \langle \lambda_v, u \rangle + \dots$ has a minimum at $\overline{u} = -\frac{\lambda_v}{\|\lambda_v\|}$ (aligned with $\delta r$, Q.E.D.)
   - 4.2 $H = 1 + \langle \lambda_v, u \rangle + \dots$ has no minimum(!!), unless we introduce a contraint like $\|u\| < u_{max}$. Then $H$ has a minimum at $\overline{u} = -\frac{\lambda_v}{\|\lambda_v\|} u_{max}$ (aligned with $\delta r$ and with $\lambda_v$, Q.E.D.)

# Indirect methods Example #1: bang-bang control I

- Assume we have the following dynamical system:

$$\begin{cases} \dot{r} = v \\ \dot{v} = F(r) + G(v) + u\frac{T}{m} \\ \dot{m} = -T/k \end{cases}$$

where the control functions are $u$ and $T$, $\|u\| = 1$, $0 \leq T \leq T_{max}$ and $k$ is a constant (exhaust velocity).

- Assume we want to minimize the propellant mass or the transfer time. In either case, $L = 0$ and the Hamiltonian is

$$H = \langle \lambda_r, v \rangle + \left\langle \lambda_v, F(r) + G(v) + u\frac{T}{m} \right\rangle + \langle \lambda_m, -T/k \rangle$$

# Indirect methods Example #1: bang-bang control II

- Let's minimize H with respect to u

$$\overline{u} = \arg\min_{\|u\|=1} H = -\frac{\lambda_v}{\|\lambda_v\|} \tag{12}$$

- Let's minimize H with respect to T

$$\overline{T} = \arg\min_{0 \leq T \leq T_{max}} H = \arg\min_{0 \leq T \leq T_{max}} \left( -\frac{T}{m} S \right)$$

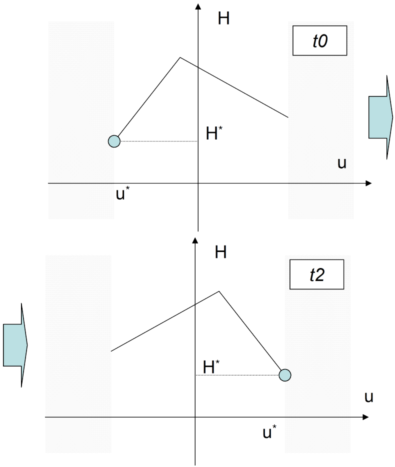
where we used Eq.(12) and introduced the *switching function*

$$S = \|\lambda_v\| + \lambda_m \frac{m}{k}$$

- Then we have the bang-bang control:

$$\begin{cases} S > 0 & \rightarrow T = T_{max} \\ S < 0 & \rightarrow T = 0 \\ S = 0 & \rightarrow T = ? \end{cases}$$

# Indirect methods Example #1: bang-bang control III




Bang-bang control, cost arcs,
H constant in time

Figure: Bang-bang control

# Direct methods Introduction

- Discretize the trajectory (transcription) and solve a large (sparse) Non-Linear Programming problem, typically with some Sequential Quadratic Programming algorithm.
- Constraints are added easily
- PMP can be used to *check* the optimality of the solution
- Robust, if coded right
- Very slow
- Limited accuracy

# Direct methods Example: Finite differences

- Not a good transcription method, but useful to illustrate the general idea
- Time, trajectory and control are discretized:

$$t_{0,} t_1, \ldots, t_N$$

$$x_i = x(t_i) \, i = 1, \ldots, N$$

$$u_i = u(t_i) \, i = 1, \ldots, N$$

- We introduce a discretized version of Eq.(1-2), using for instance Euler method

$$\tilde{J}(x_j, u_j, t_j) = \varphi(t_0, x_0, t_N, x_N) + \sum_{i=0}^{N-1} L(x_i, u_i, t_i)(t_{i+1} - t_i)$$

$$g(x_j, u_j, t_j) = x_{i+1} - x_i - f(x_i, u_i, t_i)(t_{i+1} - t_i) = 0 \qquad (13)$$

- Then we solve the parameter optimization problem of minimizing $\tilde{J}$ subject to the *algebraic* constraints Eq.(13) and boundary conditions.

# Direct methods Collocation

- Assume the states/controls can be expressed through orthogonal function. E.g., polynomials:

$$\tilde{x} = at^2 + bt + c$$

- Then the dynamic constraints are replaced by a set of algebraic collocation constraint like

$$\dot{\tilde{x}}(t_i) - f\left(\tilde{x}(t_i), \tilde{u}(t_i), t_i\right) = 0$$

$$\dot{\tilde{x}}\left(\frac{t_i + t_{i+2}}{2}\right) - f\left(\tilde{x}\left(\frac{t_i + t_{i+2}}{2}\right), \tilde{u}\left(\frac{t_i + t_{i+2}}{2}\right), \frac{t_i + t_{i+2}}{2}\right) = 0$$

$$\dot{\tilde{x}}(t_{i+1}) - f\left(\tilde{x}(t_{i+1}), \tilde{u}(t_{i+1}), t_{i+1}\right) = 0$$

- In practical application, we would use other orthogonal functions

$$\tilde{x} = \sum_{i=1}^{N} r_i(t)\tilde{x}_i$$

- Finite elements in time, also called Gauss pseudospechtral methods, replace the dynamic constraint with its weak form:

$$\int_{t0}^{t1} \left( \langle w, f \rangle + \langle \dot{w}, x \rangle \right) dt - \langle w, x \rangle \Big|_{t0}^{t1}$$

The equation is then transformed into a set of algebraic equations, where the states, control, and weight functions are discretized using orthogonal functions collocated on Gauss-Lobatto nodes

# Direct methods DMOC

- DMOC replace the dynamic constraint with the Lagrange-d'Alambert principle

$$\delta \int_{t0}^{t1} \mathscr{L}(q, \dot{q}, t) dt + \int_{t0}^{t1} u(t) \delta q(t) dt = 0$$

  The equation is then transformed into a set of algebraic equations like

$$D_2 L(q_{k-1}, q_k, h) + D_1 L(q_k, q_{k+1}, h) + u_{k-1}^{+} + u_k^{-} = 0$$