Low-Thrust Trajectories: An overview of the Q-law and other analytic techniques

Anastassios E. Petropoulos

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3^{rd} Talk in the series Optimal Control in Space Mission Design

hosted by Stefano Campagnola at ISAS/JAXA

Why Low-Thrust?

In brief: high Specific Impulse. Ideal for:

- Mass-constrained missions
- High-ΔV missions

Caveat: power source

Deep-Space Missions:

- Deep Space 1 (NASA/JPL), 1998
- Hayabusa (JAXA), May 2003
- SMART-1 (ESA), to the moon, Sept. 2003
- Dawn (NASA/JPL), 2007

Dawn will be the highest-deep-space-ΔV mission ever.

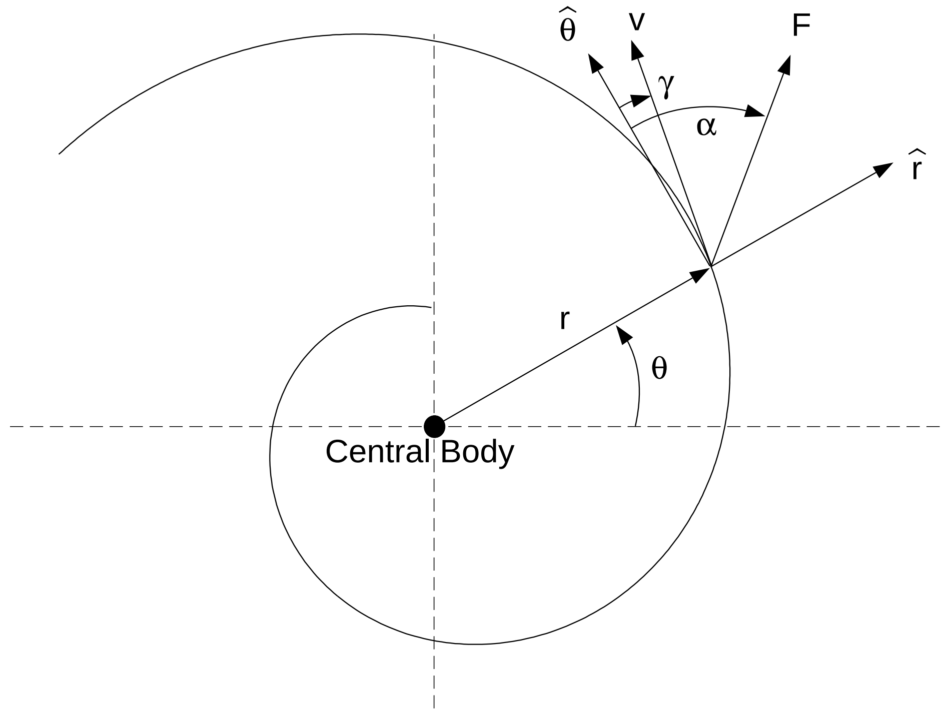
Caveat: mission design is more complicated — optimal control, not just parameter optimisation

Outline of today's discussions

- Some analytic solutions to the equations of motion of a thrusting spacecraft
- Some analytic integrals using averaging for spiralling trajectories
- Lyapunov feedback control: Lyapunov functions, Q-law, GA-Q-law.

Ask questions freely during presentation

EOMs and General Geometrical Definitions



$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = f \sin\alpha$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = f \cos\alpha$$

The Logarithmic Spiral

- **Shape equation:** $$r = r_0 \mathrm{e}^{q(\theta-\theta_0)}$$

  $$r = \text{radius}, \quad \theta = \text{polar angle}, \quad q = \text{constant}$$

- **Assuming:**

$$\left.\begin{array}{l}\text{Constant Thrust Angle}\\ \text{Constant } a = \frac{f}{\mu/r^2}\end{array}\right\} \Longrightarrow \begin{array}{rcl}\theta &=& \frac{2}{3q}\ln\left(\tilde{n}t + \mathrm{e}^{\frac{3}{2}q\theta_0}\right)\\ r &=& r_0\left(\tilde{n}t + \mathrm{e}^{\frac{3}{2}q\theta_0}\right)^{\frac{2}{3}}\end{array}$$

- **Explicit functions of time permit easy targeting**

Logarithmic Spiral: Disadvantages

- **Best suited for slowly spiraling trajectories, even if thrust assumptions removed**

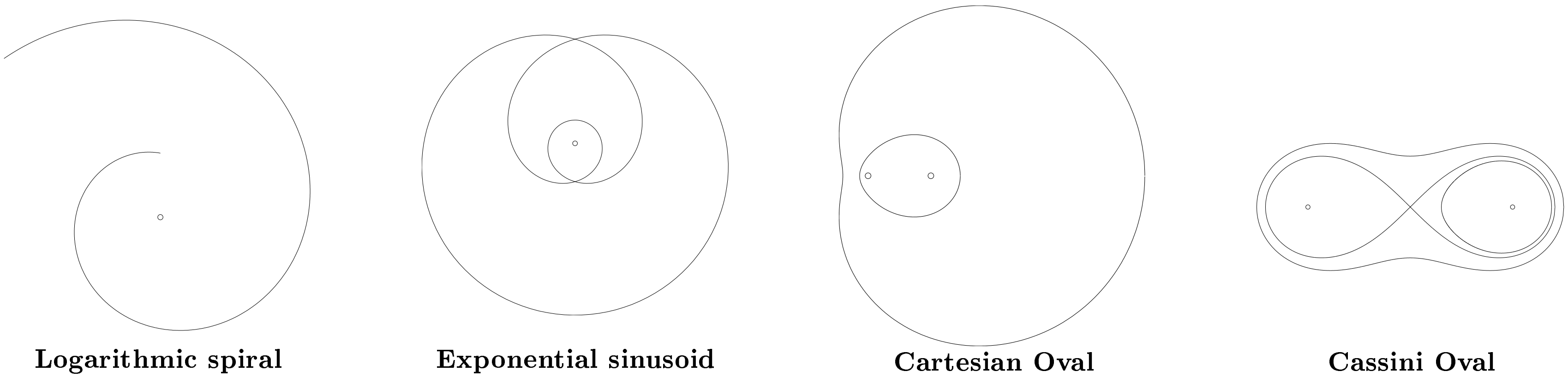
- **Poor v_∞ and TOF performance (*vs.* Hohmann)**

$$\tilde{v}_{\infty min} \propto \frac{1}{\tilde{v}_{max}\mathrm{TOF}_{min}}$$

  - **Match Hohmann v_∞, exceed Hohmann TOF**
  - **Match Hohmann TOF, exceed Hohmann v_∞**

Assumed Shape Method

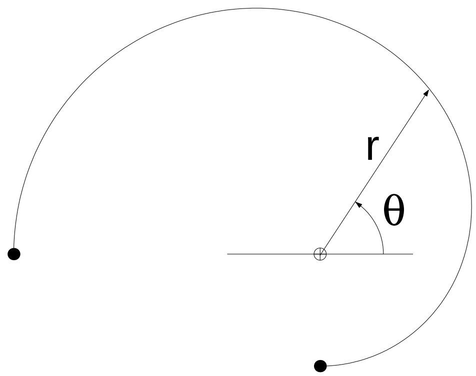
- **Trajectory shape assumption replaces conics**



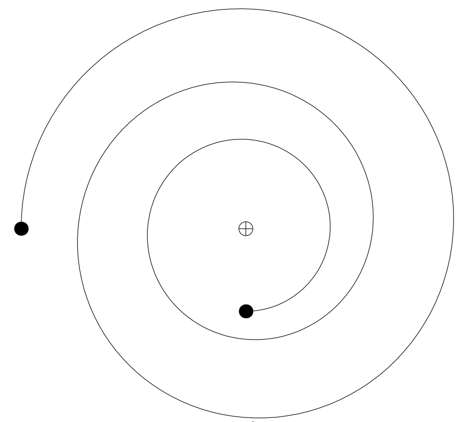
- **Shape equation + EOMs** $\Longrightarrow \dot{\theta}^2 = \left(\frac{\mu}{r^3}\right) \frac{a \sec\gamma \sin(\gamma - \alpha) + 1}{2\tan^2\gamma - r''/r + 1}$

- **Have one DOF if given state. E.g., can specify one of:**
  - Thrust angle
  - Thrust acceleration
  - Angular momentum

Exponential Sinusoid Shape



$k_2 = 2/3$



$k_2 = 2/11$

- **Exponential Sinusoid:**

$$r = k_0 \exp(k_1 \sin k_2 \theta)$$

flexibility in geometry with only 3 parameters (k_0, k_1, k_2)

- **Conic:** $r = a(1 - e^2)/(1 + e\cos\theta)$, **has 2 parameters** (a, e)

Exponential Sinusoid with Tangential Thrust

- **Makes $\dot{\theta}$ and a tractable, periodic functions of θ:**

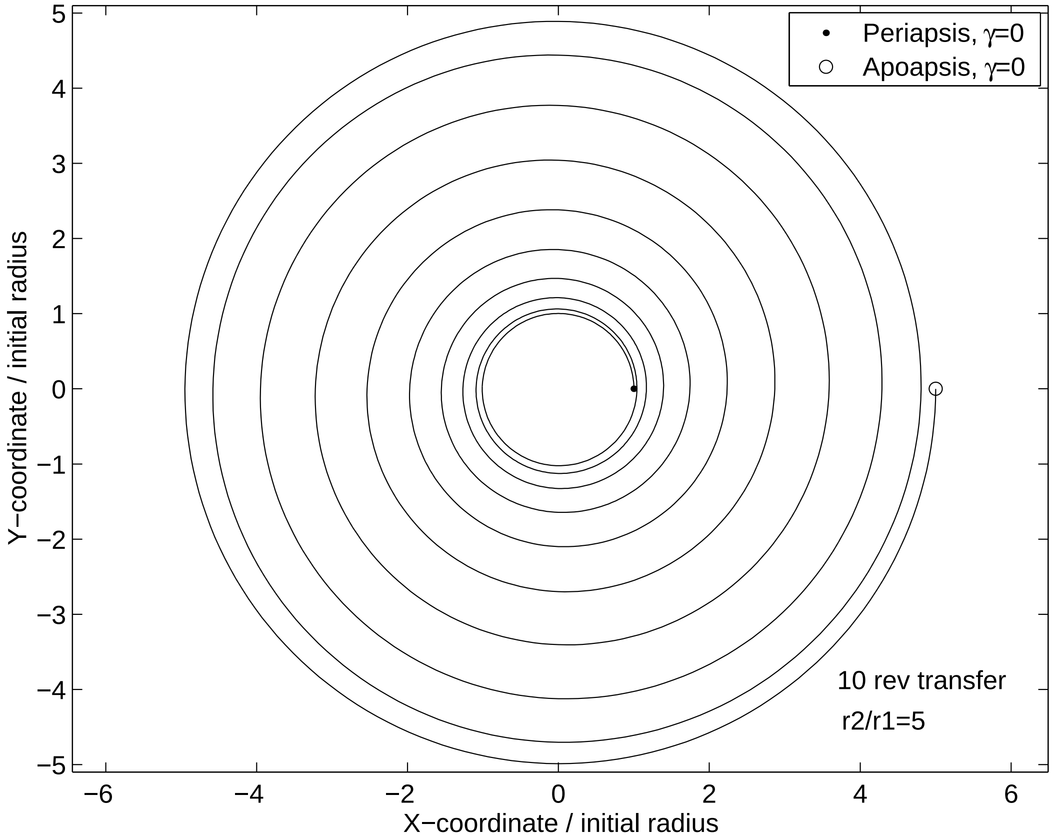
$$\dot{\theta}^2 = \left(\frac{\mu}{r^3}\right) \frac{1}{\tan^2\gamma + k_1 k_2^2 s + 1}$$

$$a = \frac{(-1)^n \tan\gamma}{2\cos\gamma} \left[\frac{1}{\tan^2\gamma + k_1 k_2^2 s + 1} - \frac{k_2^2(1 - 2k_1 s)}{\left(\tan^2\gamma + k_1 k_2^2 s + 1\right)^2}\right]$$

- **TOF available through quadrature**

$$t = \int \dot{\theta}^{-1} \, \mathrm{d}\theta$$

Exp Sine Multi-Rev Orbit Transfer

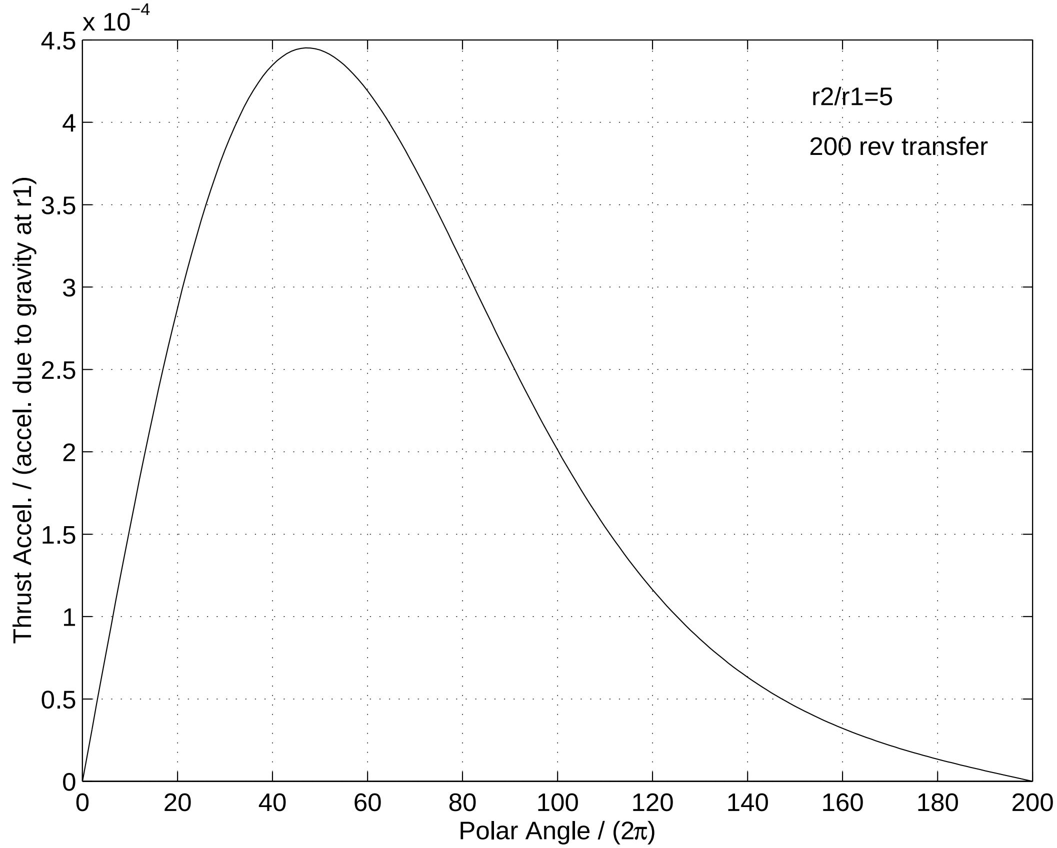


$$\Delta V_1 \approx v_{c1} \frac{\ln(r_2/r_1)}{16N^2}$$

$$\Delta V_2 \approx v_{c2} \frac{\ln(r_2/r_1)}{16N^2}$$

$$\Delta V_a \approx v_{c1} - v_{c2}$$

Exp Sine Multi-Rev Orbit Transfer: Thrust

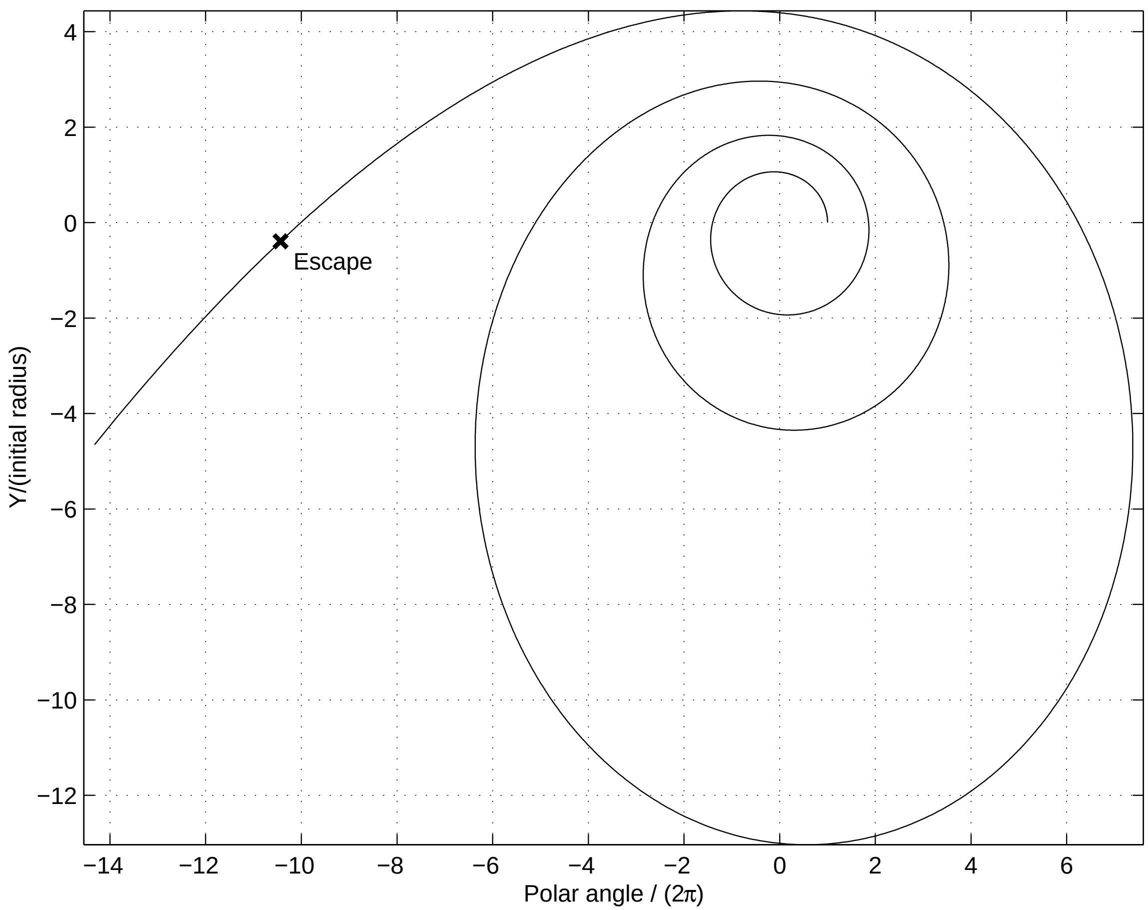


Pinkham's Spiral

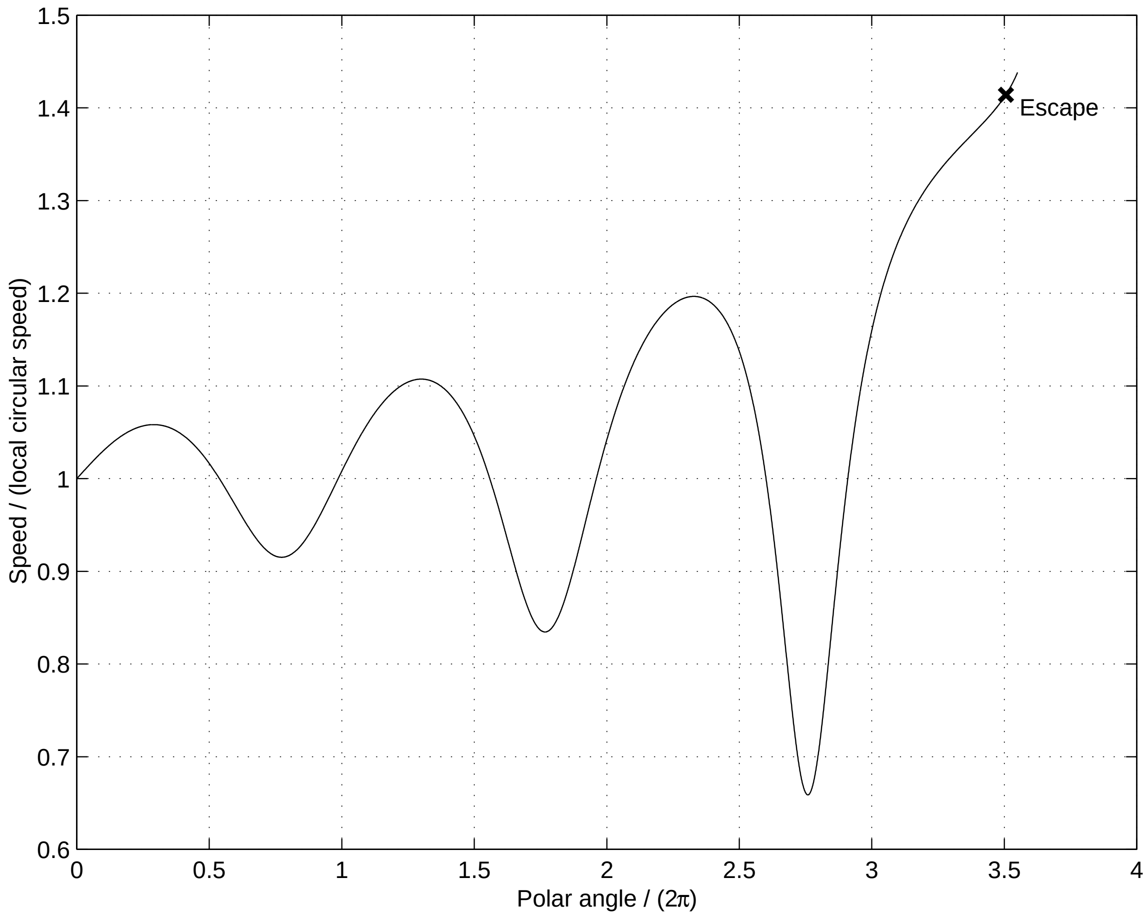
$$
\begin{aligned}
f_r &= \frac{q\sqrt{\mu p}}{2r^2} v_r \\
f_\theta &= \frac{q\sqrt{\mu p}}{2r^2} v_\theta \\
p &= p_s \mathrm{e}^{q\theta} \\
r &= \frac{p(1+q^2)}{1+\mathrm{e}^{q\theta}(1+q^2)k\cos(\theta-\omega)}
\end{aligned}
$$

- **Four constants: p_s (semilatus rectum scale), q, k, ω**
- **Logarithmic spiral variant**

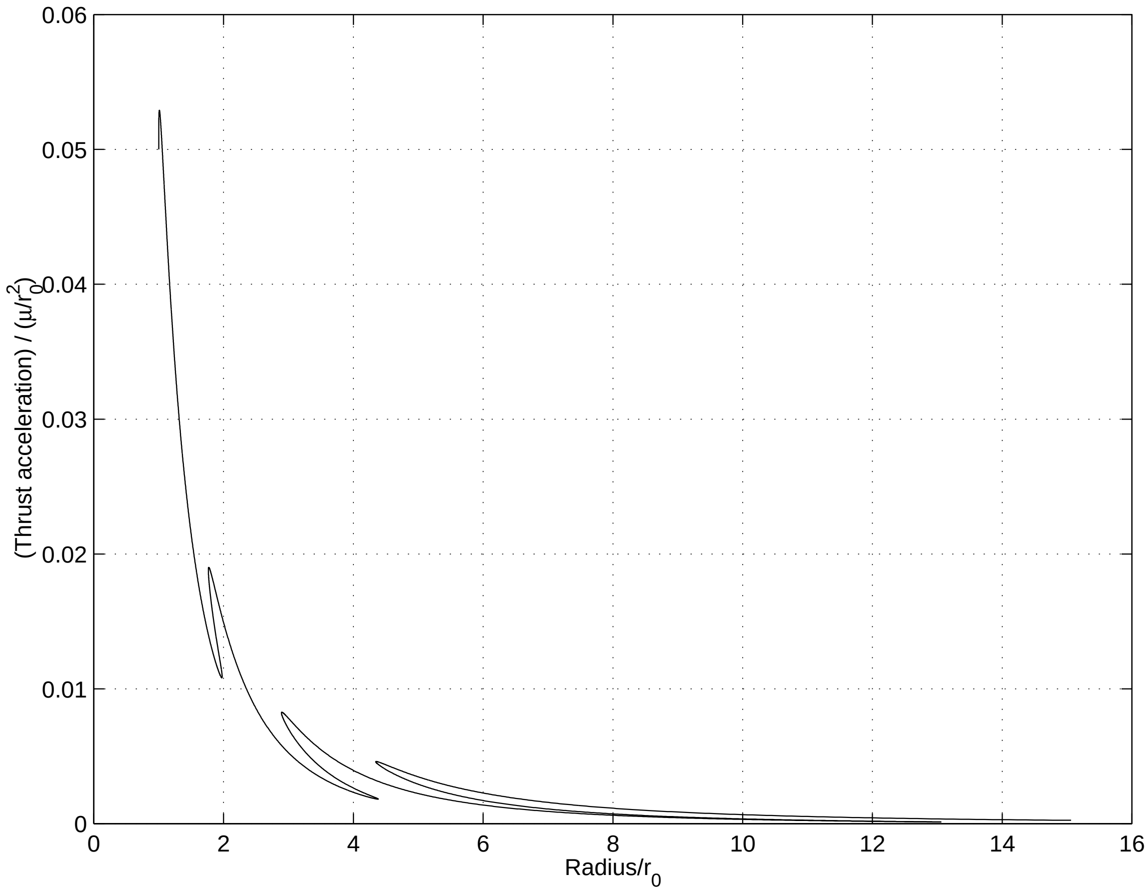
Pinkham's Spiral: Shape



Pinkham's Spiral: Speed



Pinkham's Spiral: Thrust

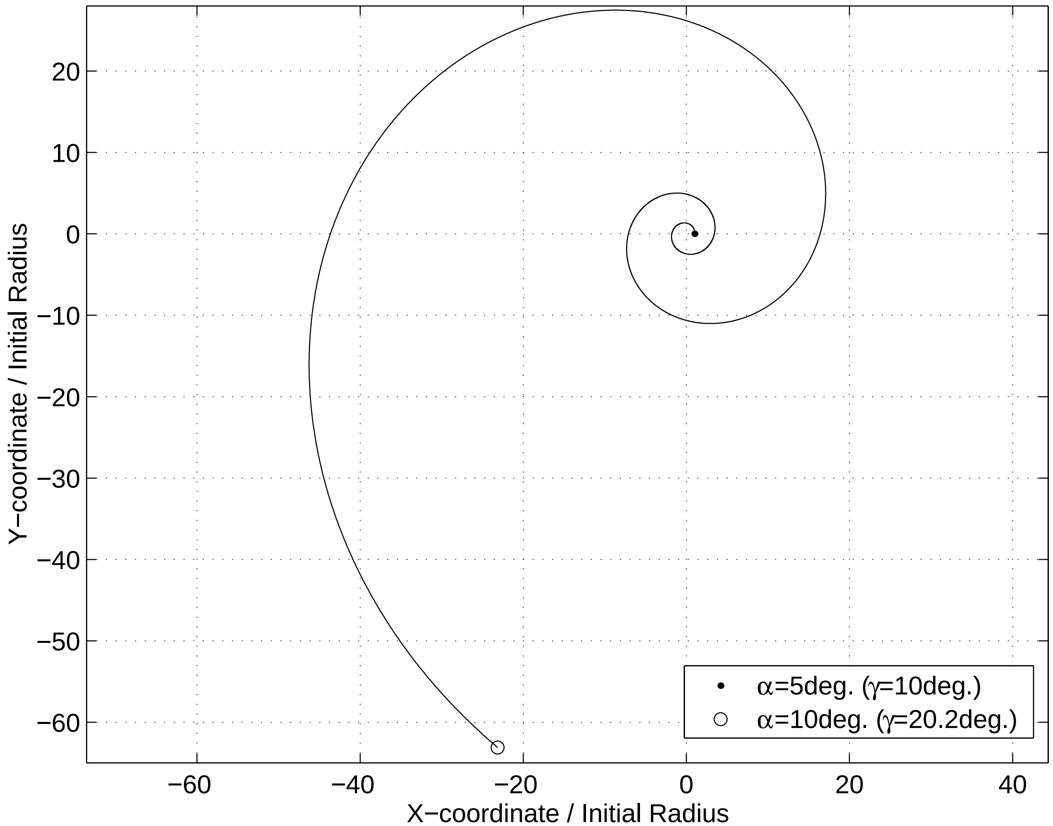


Markopoulos' Keplerian Thrust

$$
\begin{aligned}
f_\theta &= \frac{Q}{r} \\
f_r &= \frac{Q\dot{r} + 2\dot{Q}r}{h} \\
\tilde{r} &= \frac{\tilde{h}^2}{1 - B^2} \cdot (1 - B\cos E) \\
\tan\frac{\theta}{2} &= \sqrt{\frac{1+B}{1-B}}\tan\frac{E}{2} \\
h &= h_0 + \textstyle\int Q\mathrm{d}t \\
\int \frac{1}{\tilde{h}^3}\mathrm{d}\tilde{t} &= \frac{1}{(1-B^2)^{3/2}} \cdot [E - E_0 - B(\sin E - \sin E_0)]
\end{aligned}
$$

- **E.g.,** $Q = Q_0 \implies$ **tangential thrust** $\implies h = h(E)$ **explicitly**

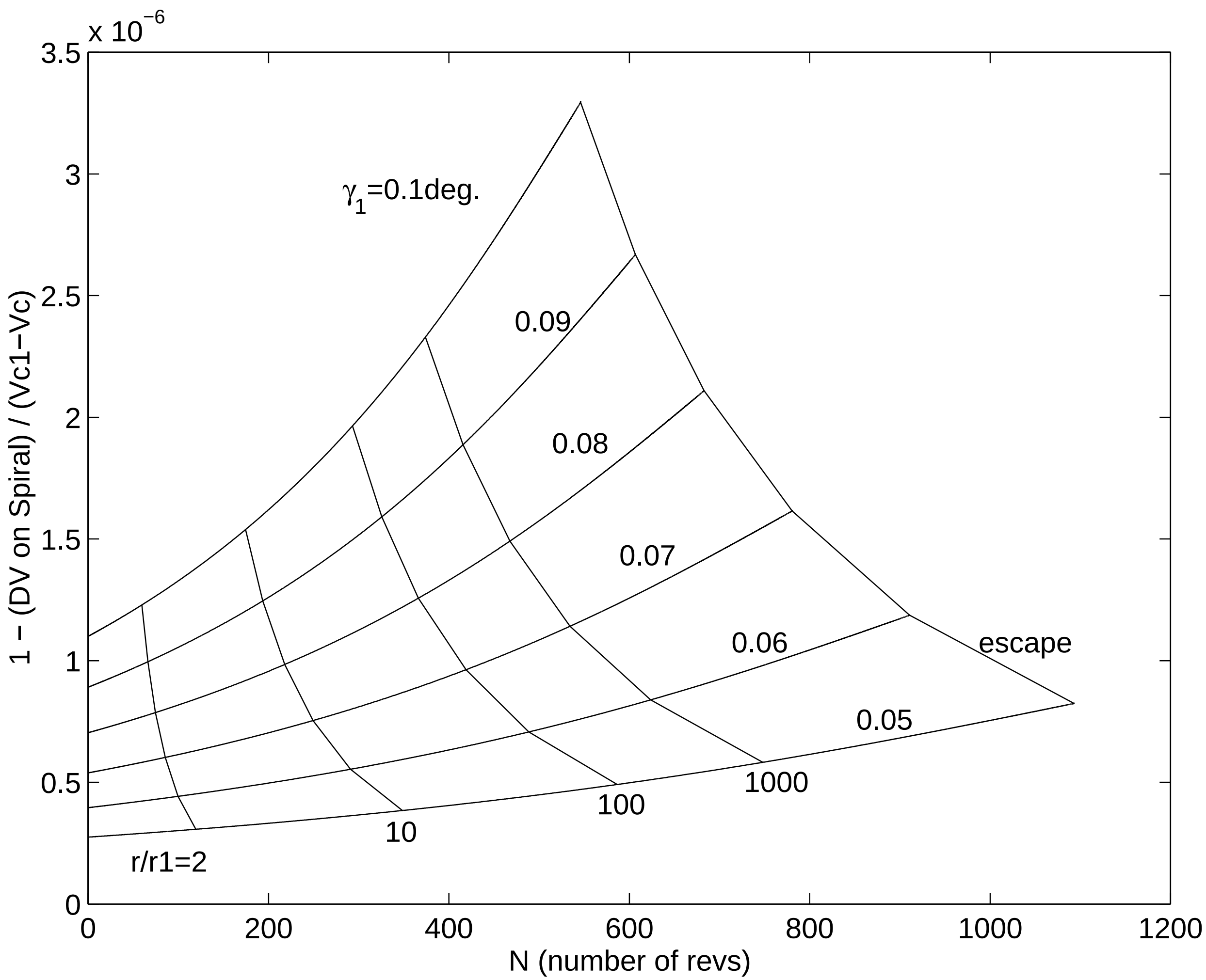
Lawden's Spiral



$$r = \frac{r_s \sin^6 \alpha}{1 - 3\sin^2 \alpha}$$

$$\theta = \theta_0 - 4\alpha - 3\cot\alpha$$

ΔV on Lawden's Spiral



Other Solutions

- Forbes' spiral, Moeckel's spiral
- Forbes' Keplerian observation: $\dfrac{f}{\dot{r}\sin\gamma} = \dfrac{\ddot{A}}{\dot{A}}$
- Bishop and Azimov spiral
- Stark problem: point mass gravity plus constant inertial acceleration
- Newer shape-based approaches by:
  - Wall, Conway
  - Vasile, de Pascale, Cassoto
  - Abdelkhalik, Taheri

Continuous Tangential Thrust and Averaging for Spirals

- Motivated by spiral escape and capture trajectories from or to orbits of arbitrary eccentricity.
- Assume $\frac{1}{r^2}$ gravity to simplify the analytical approach
- In practice, thrust is normally constant and small, and I_{sp} high
  - Thrust acceleration nearly constant over spiral
  - Orbital elements vary only slightly over one revolution

Numerically examine the case of escape from Geostationary Transfer Orbit (GTO) as a springboard for the analytical relations

Variation in escape time, tangential thrust

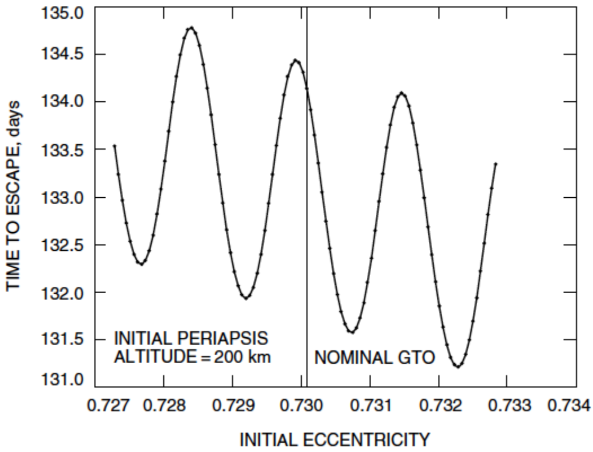


Fig. 1. Oscillation in escape time using tangential thrust for various initial Earth orbits near the nominal GTO, with thrust starting at periapsis.

GTO: 200km x 35,786km altitude; Thrust $= 465$mN, $I_{sp} = 3100$s, $m_0 = 1500$kg

Evolution of the eccentricity, tangential thrust

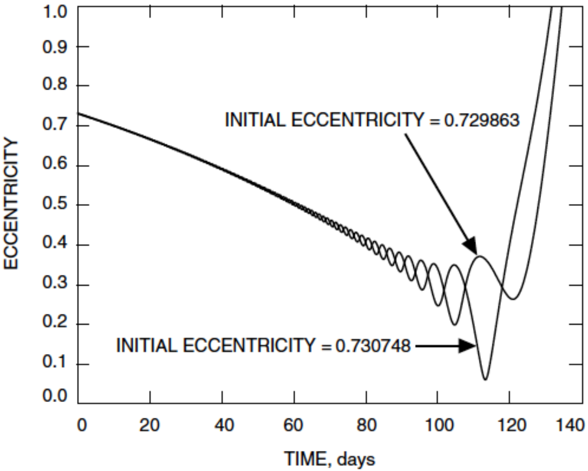


Fig. 3. Evolution of eccentricity with time for tangential thrust for orbits of the two different initial eccentricities that give the local maximum and minimum escape times immediately adjacent to the nominal GTO line in Fig. 1.

Variation in the minimum achieved eccentricity, tangential thrust

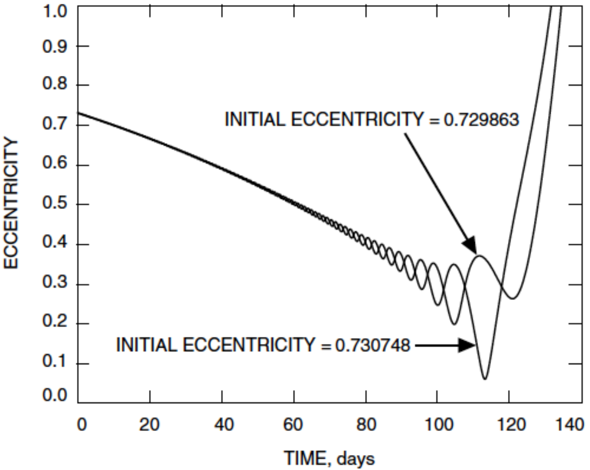


Fig. 3. Evolution of eccentricity with time for tangential thrust for orbits of the two different initial eccentricities that give the local maximum and minimum escape times immediately adjacent to the nominal GTO line in Fig. 1.

Averaging Analysis, Energy I

Assume

- Single-point-mass gravity
- Planar thrust acceleration with constant tangential component

Using specific orbital energy provides more insights (than a)

$$\mathcal{E} = -\frac{\mu}{2a}$$

From elementary orbit mechanics

$$\mathrm{d}\mathcal{E} = f_t \mathrm{d}s$$

$f_t =$ tangential component of thrust acceleration
$s =$ trajectory arc length (inertial)

This reduces immediately to the customary

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2a^2 v}{\mu} f_t$$

Averaging Analysis, Energy II

However, stick with energy, and use geometry. Ellipse perimeter is

$$S_p = 4aE_I(e), \quad \text{where} \quad E_I(e) = \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \sin^2\theta}\, \mathrm{d}\theta$$

$E_I(e)$ = complete elliptic integral, 2^{nd} kind, modulus e
And so we can integrate:

$$\Delta\mathcal{E}^{2\pi} \approx 4f_t a E_I(e)$$

And so, average rate of change of energy is

$$\overline{\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}}^{t,2\pi} \approx \frac{2f_t}{\pi}\sqrt{\frac{\mu}{a}}E_I(e)$$

RHS depends only on $\mathcal{E}(a)$ and e and is separable.

Averaging Analysis, Eccentricity and Energy I

From Gauss's variational equations for eccentricity and eccentric anomaly

$$\begin{aligned}\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1}{v}\left[2\left(e+\cos\theta\right)f_t + \frac{r}{a}f_n\sin\theta\right] \\ \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{na}{r} - \frac{1}{ebv}\left[2af_t\sin\theta - r\left(e+\cos\theta\right)f_n\right]\end{aligned}$$

Then, for e "not too small", and f_t, f_n not too big,

$$\frac{\mathrm{d}E}{\mathrm{d}t} \approx \frac{na}{r}$$

and

$$\frac{\mathrm{d}e}{\mathrm{d}E} \approx \frac{r}{na}\frac{\mathrm{d}e}{\mathrm{d}t}$$

RHS is integrablt wrt E, after many manipulations

Averaging Analysis, Eccentricity and Energy II

For one full revolution in E, the integral reduces to

$$\overline{\Delta e}^{2\pi} \approx -\frac{8f_t(1-e^2)}{e\mu/a^2}[K(e) - E_I(e)]$$

where the RHS is independent of f_n, and

$$K(e) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1 - e^2\sin^2\theta}}$$

$K(e) =$ complete elliptic integral, first kind, modulus e.
And so we have

$$\overline{\frac{\mathrm{d}e}{\mathrm{d}t}}^{t,2\pi} \approx -\frac{4f_t(1-e^2)}{\pi e}\sqrt{\frac{\mu}{a}}[K(e) - E_I(e)]$$

$$\overline{\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}e}}^{t,2\pi} = \frac{\overline{\Delta\mathcal{E}}^{2\pi}}{\overline{\Delta e}^{2\pi}} \approx -\frac{\mu}{2a} \cdot \frac{eE_I(e)}{(1-e^2)[K(e) - E_I(e)]}$$

Averaging Analysis, Eccentricity and Energy III

Remarkably, the $\overline{\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}e}}^{t,2\pi}$ equation

- doesn't depend on f_t or f_n
- is integrable after a series of manipulations

$$\frac{\mathcal{E}}{\mathcal{E}_0} \approx \frac{[K(e) - E_I(e)]}{[K(e_0) - E_I(e_0)]}$$

The depenence on time is determined by substituting the above into the $\overline{\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}}^{t,2\pi}$ equation and integrating using a number of of integration variable changes, series expansions and manipulations of the elliptic integrals. Also, rather than using time, ΔV is used as a time-like variable:

$$\Delta V = \int_0^t f_t \mathrm{d}t$$

If f_t is constant, then $t = \Delta V / f_t$.

If thrust and I_{sp} are constant, $t = \frac{c}{f_t}(1 - \mathrm{e}^{-\frac{\Delta V}{c}})$

Averaging Analysis, Integral for e 'large', f 'small'

$$
\begin{aligned}
\Delta V \;\approx\; & \pi\sqrt{\frac{-\mathcal{E}_{e0}}{2}}\left\{\frac{2}{\pi}\left[\left(\frac{\mathcal{E}_0}{\mathcal{E}_{e0}}\right)^{\frac{1}{2}}-\left(\frac{\mathcal{E}}{\mathcal{E}_{e0}}\right)^{\frac{1}{2}}\right]+\frac{2}{3\pi^2}\left[\left(\frac{\mathcal{E}_0}{\mathcal{E}_{e0}}\right)^{\frac{3}{2}}-\left(\frac{\mathcal{E}}{\mathcal{E}_{e0}}\right)^{\frac{3}{2}}\right]\right.\\
& +\frac{1}{10\pi^3}\left[\left(\frac{\mathcal{E}_0}{\mathcal{E}_{e0}}\right)^{\frac{5}{2}}-\left(\frac{\mathcal{E}}{\mathcal{E}_{e0}}\right)^{\frac{5}{2}}\right]-\frac{3}{14\pi^4}\left[\left(\frac{\mathcal{E}_0}{\mathcal{E}_{e0}}\right)^{\frac{7}{2}}-\left(\frac{\mathcal{E}}{\mathcal{E}_{e0}}\right)^{\frac{7}{2}}\right]\\
& \left.-\frac{107}{288\pi^5}\left[\left(\frac{\mathcal{E}_0}{\mathcal{E}_{e0}}\right)^{\frac{9}{2}}-\left(\frac{\mathcal{E}}{\mathcal{E}_{e0}}\right)^{\frac{9}{2}}\right]\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta V \;\approx\; & \sqrt{\frac{-\mathcal{E}_{e0}\pi}{2}}\left[(e_0-e)+\frac{13}{48}(e_0^3-e^3)+\frac{383}{2560}(e_0^5-e^5)\right.\\
& \left.+\frac{5833}{57344}(e_0^7-e^7)+\frac{43649}{589824}(e_0^9-e^9)\right]
\end{aligned}
$$

where

$$
\mathcal{E}_{e0}=\frac{\mathcal{E}_0}{[K(e_0)-E_I(e_0)]}
$$

Averaging Analysis, Integral for e 'small', f 'small'

$$\frac{e}{e_0} \approx \frac{\mathcal{E}_0^2}{\mathcal{E}^2} = \frac{v_{c0}^4}{v_c^4}$$

$$\begin{aligned}
\Delta V \;\; & \approx \;\; v_{c0} \left\{ \left(1 - \frac{v_c}{v_{c0}}\right) + \frac{e_0^2}{28}\left[\left(\frac{v_{c0}}{v_c}\right)^7 - 1\right] + \frac{7e_0^4}{960}\left[\left(\frac{v_{c0}}{v_c}\right)^{15} - 1\right] \right. \\
& \left. \quad + \frac{15e_0^6}{5888}\left[\left(\frac{v_{c0}}{v_c}\right)^{23} - 1\right] + \frac{723e_0^8}{507904}\left[\left(\frac{v_{c0}}{v_c}\right)^{31} - 1\right] \right\} \\
& \approx \;\; v_{c0} \left\{ \left[1 - \left(\frac{e_0}{e}\right)^{\frac{1}{4}}\right] + \frac{1}{28}\left[e^2\left(\frac{e_0}{e}\right)^{\frac{1}{4}} - e_0^2\right] + \frac{7}{960}\left[e^4\left(\frac{e_0}{e}\right)^{\frac{1}{4}} - e_0^4\right] \right. \\
& \left. \quad + \frac{15}{5888}\left[e^6\left(\frac{e_0}{e}\right)^{\frac{1}{4}} - e_0^6\right] + \frac{723}{507904}\left[e^8\left(\frac{e_0}{e}\right)^{\frac{1}{4}} - e_0^8\right] \right\}
\end{aligned}$$

Averaging Analysis, Integral Regions: *ef*-plot

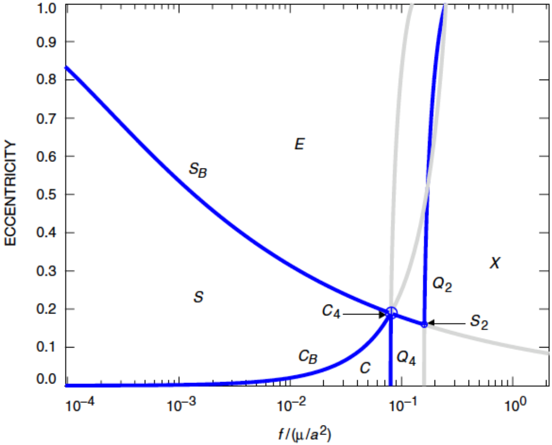


Fig. 6. Regions of the ef-plot (eccentricity versus thrust-to-semimajor-weight ratio): circular region, C; semi-elliptic region, S; pure elliptic region, E; escape region, X; circularization boundary, C_B; semi-elliptic boundary, S_B; four and two quarter revolutions to escape curves, Q_4, Q_2; terminus of C_B on Q_4, C_4; and terminus of S_B on Q_2, S_2. The light gray parts of curves are not active boundaries.

Averaging Analysis, *ef*-plot numerical examples

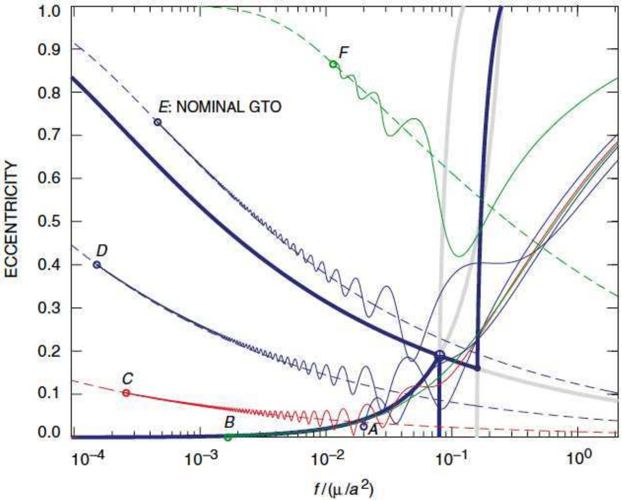


Fig. 9. Comparisons of spirals computed by numerical integration and by the averaging analysis for initial orbits lying in various regions of the ef-plot.

Averaging Analysis, GTO numerical results I

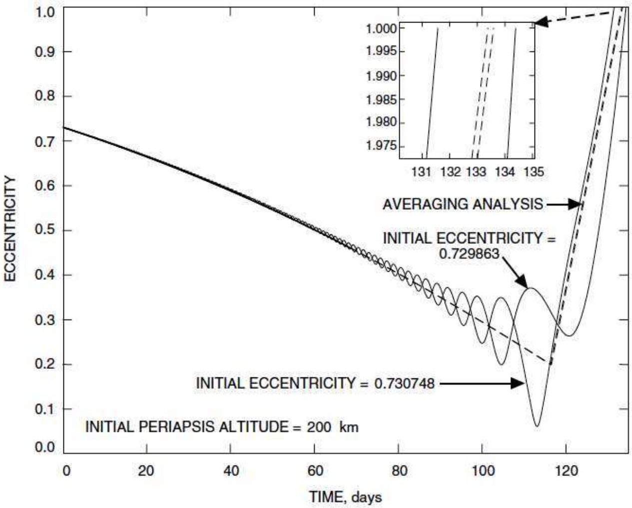


Fig. 10. Comparison of eccentricity as a function of time for the two near-GTO cases shown in Fig. 3, computed by numerical integration and by the averaging analysis.

Averaging Analysis, GTO numerical results II

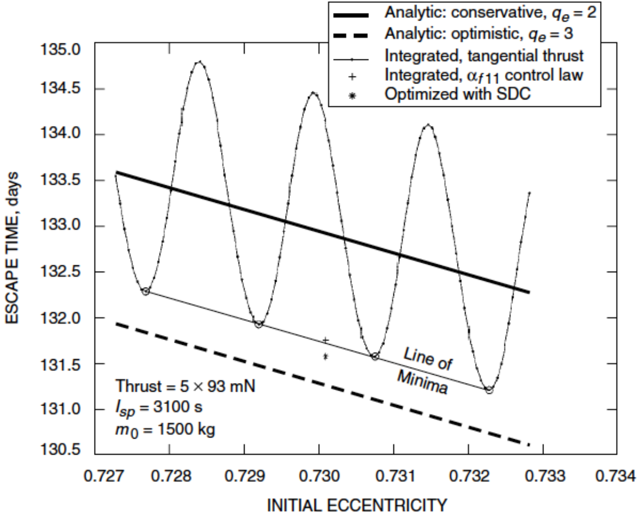


Fig. 7. Comparison of escape times computed by numerical integration and by the averaging analysis for orbits near GTO.

Good agreement with minimum-time escape (free thrust direction)

Lyapunov feedback laws and Q-law

- Rationale for Q-law
- What makes a good Lyapunov function
- Refinements to Q-law
- GA-Q-law
- Comparison with optimal solutions

Gauss's Form of the Variational Equations

$$\frac{d\Omega}{dt} = \frac{r\sin(\theta+\omega)}{h\sin i} f_h$$

$$\frac{di}{dt} = \frac{r\cos(\theta+\omega)}{h} f_h$$

$$\frac{d\omega}{dt} = \frac{1}{eh}\left[-p\cos\theta f_r + (p+r)\sin\theta f_\theta\right] - \frac{r\sin(\theta+\omega)\cos i}{h\sin i} f_h$$

$$\frac{da}{dt} = \frac{2a^2}{h}\left(e\sin\theta f_r + \frac{p}{r} f_\theta\right)$$

$$\frac{de}{dt} = \frac{1}{h}\left\{p\sin\theta f_r + \left[(p+r)\cos\theta + re\right] f_\theta\right\}$$

$$\frac{d\theta}{dt} = \frac{h}{r^2} + \frac{1}{eh}\left[p\cos\theta f_r - (p+r)\sin\theta f_\theta\right]$$

Analytic Insights From the Variational Equations

With $\text{œ} = \boldsymbol{\Omega}, \boldsymbol{i}, \boldsymbol{\omega}, \boldsymbol{a}$, or $\boldsymbol{e}$:

$$\boldsymbol{\alpha}_{\text{œ}\mathrm{x}} = \boldsymbol{f}(\text{œ}, \boldsymbol{\theta}) \quad : \textbf{ thrust angle giving maximum } \dot{\text{œ}}$$

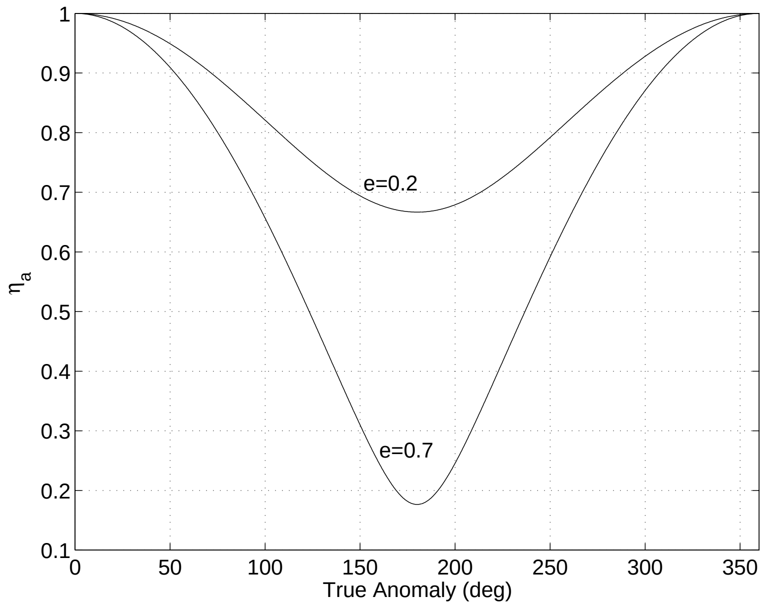
$$\dot{\text{œ}}_{\mathrm{x}} = \boldsymbol{f}(\text{œ}, \boldsymbol{\theta}) \quad : \textbf{ max } \dot{\text{œ}} \textbf{ over thrust angle}$$

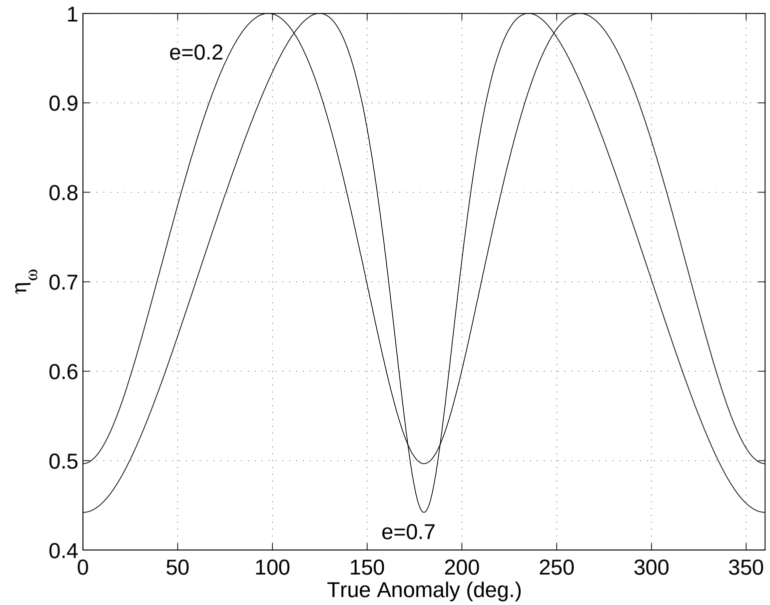
$$\dot{\text{œ}}_{\mathrm{xx}} = \boldsymbol{f}(\text{œ}) \quad : \textbf{ max over } \boldsymbol{\theta} \textbf{ of max over thrust angle of } \dot{\text{œ}}$$

$$\boldsymbol{\theta}_{\text{œ}\mathrm{xx}} = \boldsymbol{f}(\text{œ}) \quad : \textbf{ true anomaly giving } \dot{\text{œ}}_{\mathrm{xx}}$$

Effectivity Factor

$$\boldsymbol{\eta} = \frac{\dot{\text{\ae}}_{\mathbf{x}}}{\dot{\text{\ae}}_{\mathbf{xx}}}$$





Similar analytically-based plots available for all elements

Time-to-Go Concept

Coast periods and thrust direction determined based on:

- **Effectivity factor**

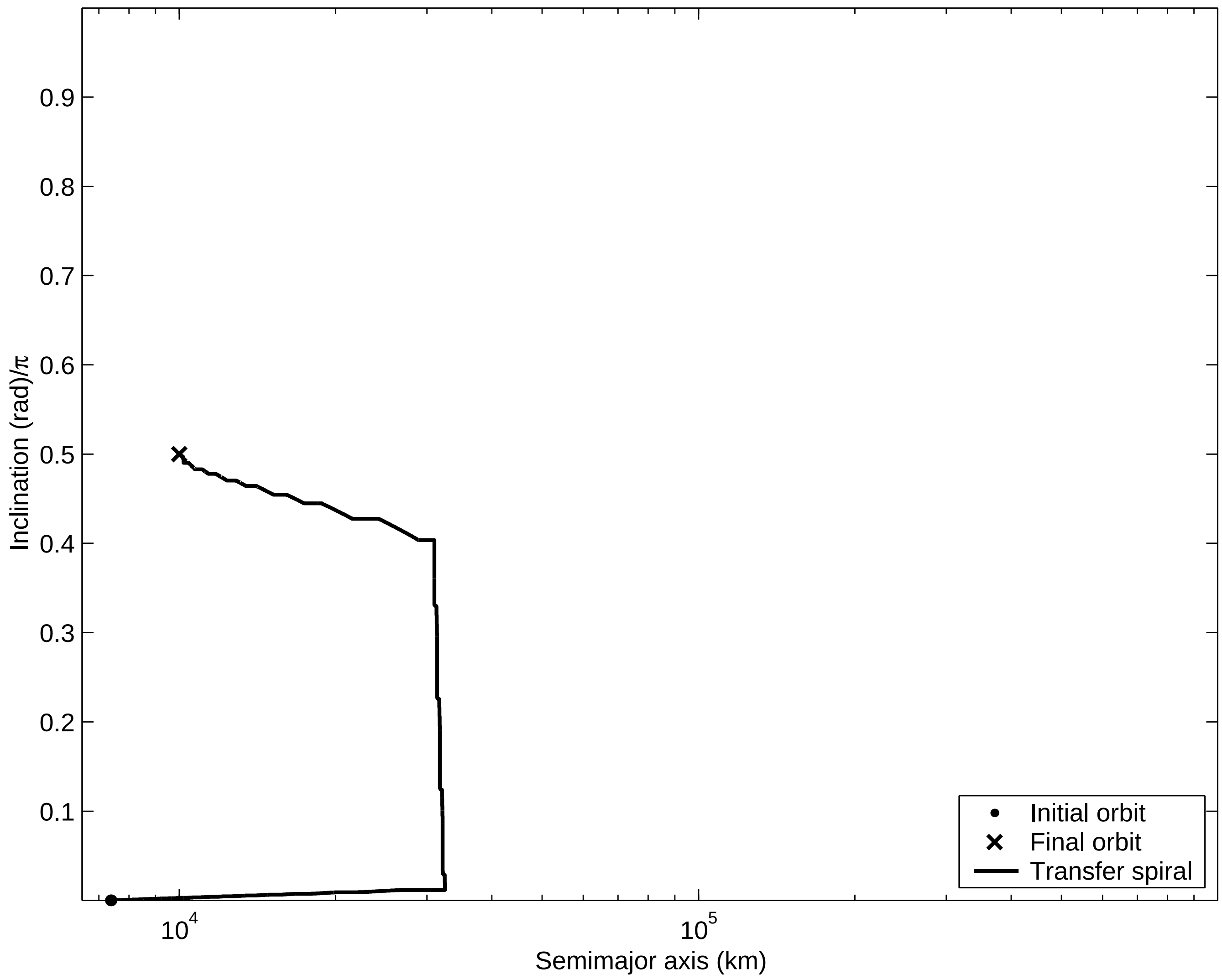
$$\boldsymbol{\eta} = \frac{\dot{æ}_{\mathrm{x}}}{\dot{æ}_{\mathrm{xx}}}$$

- **Sacrificial overshoot in one element to assist changing another if**

$$\frac{|æ_1 - æ_{1\,T}|}{\dot{æ}_{1\,\mathrm{xx}}} \gg \frac{|æ_2 - æ_{2\,T}|}{\dot{æ}_{2\,\mathrm{xx}}}$$

- **Otherwise, thrust to give equal time-to-go for each orbit element**

Time-to-Go Transfer in a and i



Proximity-Quotient Concept

- **Proximity quotient indicates how "close" we are to the target orbit:**

$$\boldsymbol{Q} = \sum_{\text{œ}} \boldsymbol{W}_{\text{œ}} \left(\frac{\text{œ} - \text{œ}_{\boldsymbol{T}}}{\dot{\text{œ}}_{\text{xx}}} \right)^{\mathbf{2}}$$

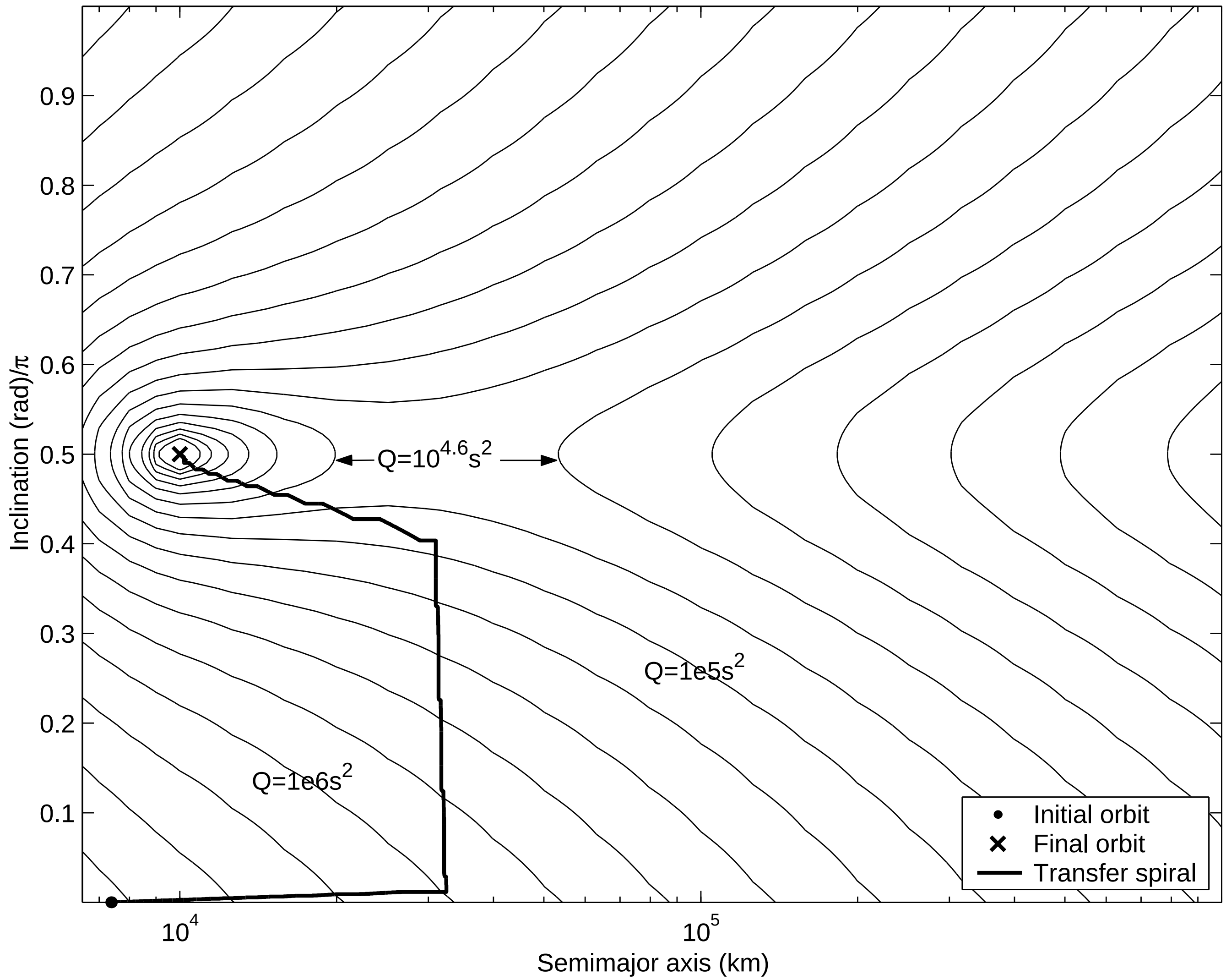
- **How quickly can we get "closer":**

$$\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}\boldsymbol{t}} = \sum_{\text{œ}} \frac{\boldsymbol{\partial Q}}{\boldsymbol{\partial}\,\text{œ}}\, \dot{\text{œ}}$$

- **How effectively can we now get "closer," compared to the optimum location on the osculating orbit to get "closer":**

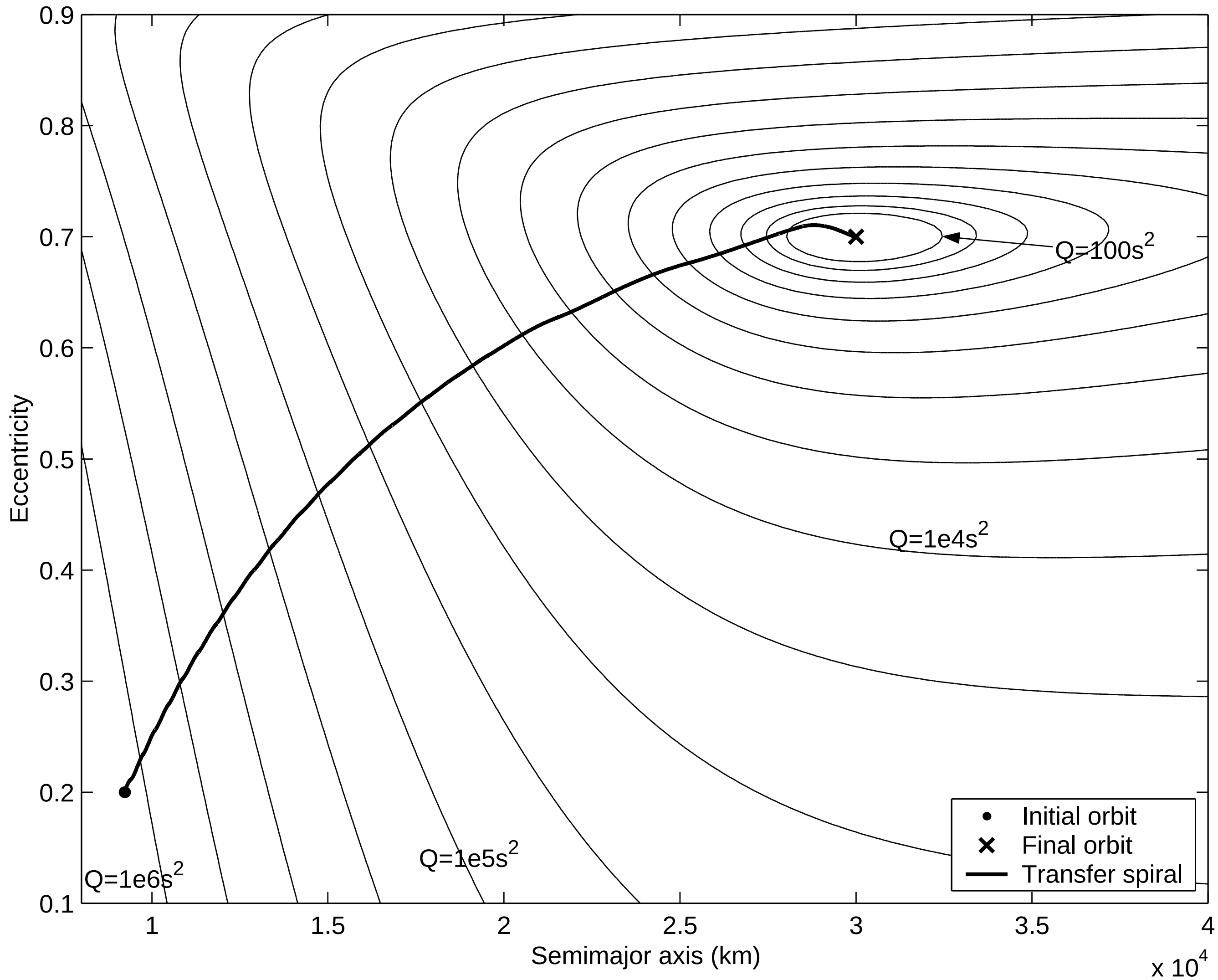
$$\boldsymbol{\eta} = \frac{\dot{\boldsymbol{Q}}_{\mathbf{n}}}{\dot{\boldsymbol{Q}}_{\mathbf{nn}}}$$

Q Contour Plot

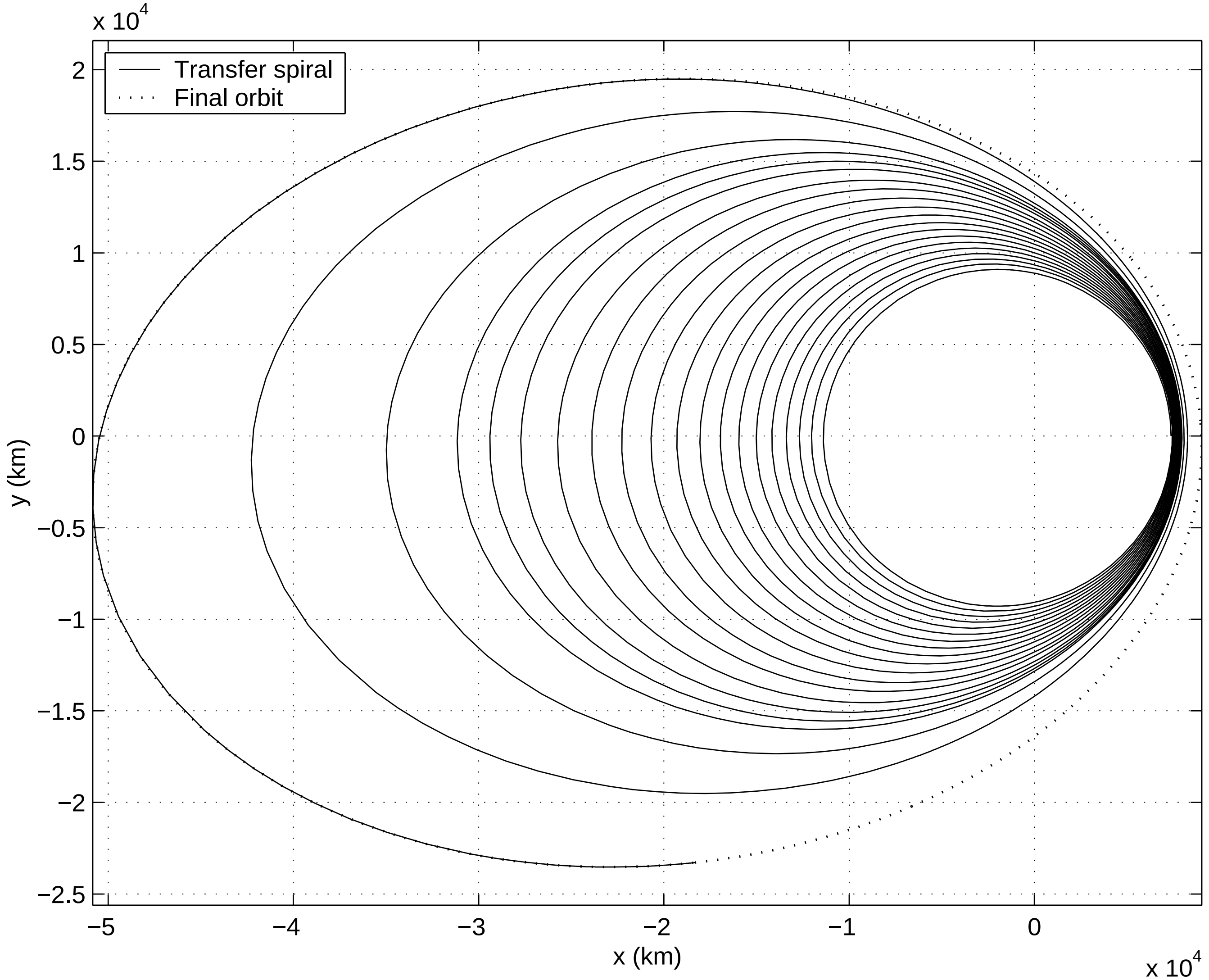


Q Contour Plot: Another Example

Transfer in a and e

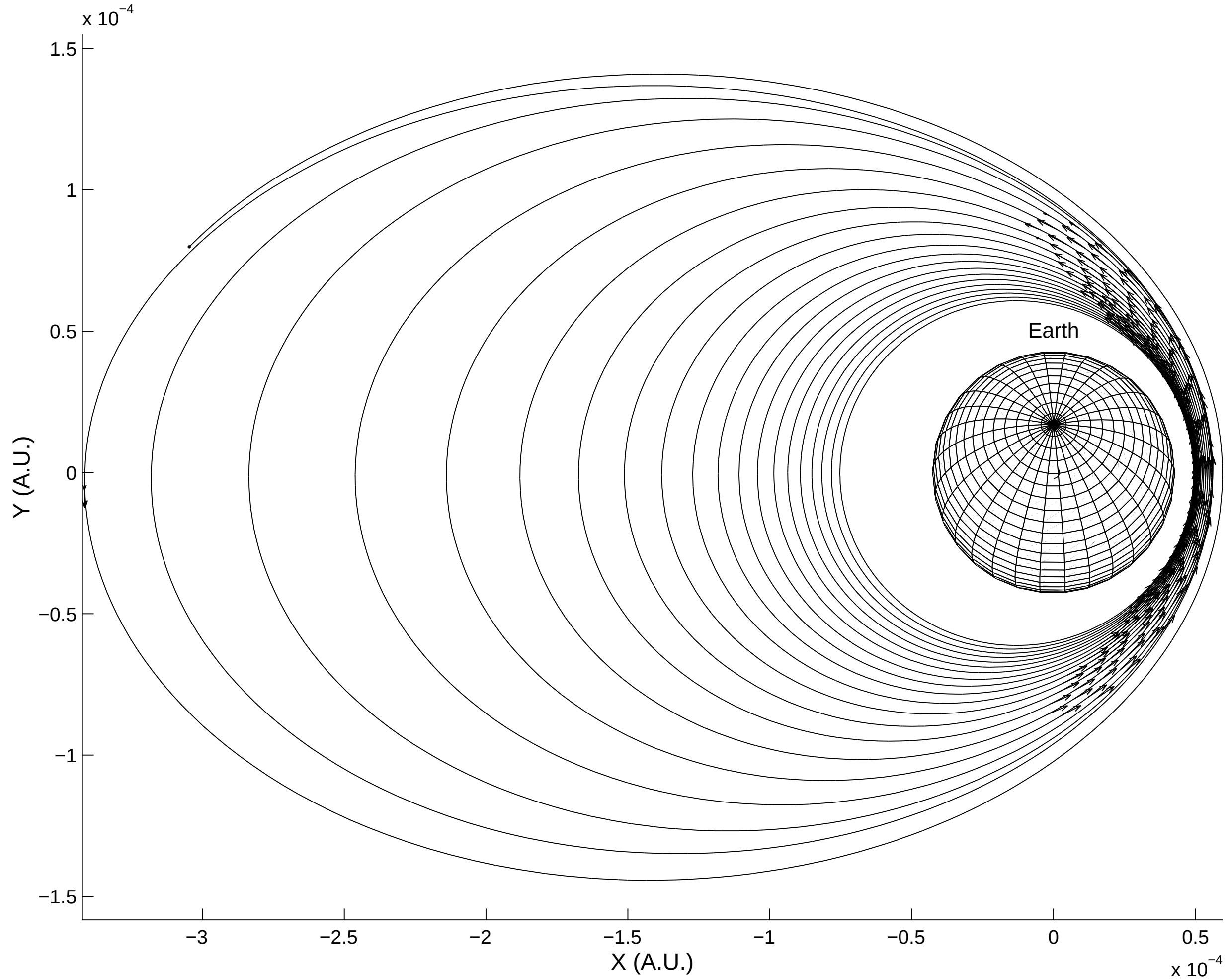


Transfer in $\boldsymbol{a}$–$\boldsymbol{e}$ Based on Proximity Quotient



$\boldsymbol{\eta_{\mathrm{cut}} = 0.6}$ **(determines number of revs and flight time)**

Propellant-Optimal Transfer in a–e



One of many local minima for fixed flight time

Comparison of Results for a–e Transfers

	Time-to-go law	Proximity-quotient law	SDC-optimised
Initial mass, kg	300	300	300
Final mass, kg	278.42	280.96	281.49
Transfer time, days	4.923	4.50	4.433[a]
Number of revolutions	19.84	19.63	19.50[a]

[a]SDC algorithm was given a fixed flight time of 4.923 days, but it chose to coast after 4.433 days flight time, performing almost one full further coast revolution to end at 20.45 revs.

Optimal Instantaneous Changes in Elements, I

$$\dot{a}_{\mathrm{xx}} = 2f\sqrt{\frac{a^3(1+e)}{\mu(1-e)}}$$

$$\dot{e}_{\mathrm{xx}} = \frac{2pf}{h}$$

$$\dot{i}_{\mathrm{xx}} = \frac{pf}{h\left(\sqrt{1-e^2\sin^2\omega} - e|\cos\omega|\right)}$$

$$\dot{\omega}_{\mathrm{xx}} = \frac{f}{eh}\sqrt{p^2\cos^2\theta_{\mathrm{xx}} + (p+r_{\mathrm{xx}})^2\sin^2\theta_{\mathrm{xx}}}$$

$$\dot{\Omega}_{\mathrm{xx}} = \frac{pf}{h\sin i\,\left(\sqrt{1-e^2\cos^2\omega} - e|\sin\omega|\right)}$$

Optimal Instantaneous Changes in Elements, II

$$
\begin{aligned}
\cos\theta_{\mathrm{xx}} &= \left[\frac{1-e^2}{2e^3} + \sqrt{\frac{1}{4}\left(\frac{1-e^2}{e^3}\right)^2 + \frac{1}{27}}\right]^{\frac{1}{3}} \\
&\quad - \left[-\frac{1-e^2}{2e^3} + \sqrt{\frac{1}{4}\left(\frac{1-e^2}{e^3}\right)^2 + \frac{1}{27}}\right]^{\frac{1}{3}} - \frac{1}{e} \\
r_{\mathrm{xx}} &= \frac{p}{1 + e\cos\theta_{\mathrm{xx}}}
\end{aligned}
$$

A word on propagation

- Q-law formulated in classical orbital elements
- Propagation of the equations of motion can be done in any element set and with any dynamics. At each integration step, the osculating classical elements and the parameters of the Q-law must be passed to the Q-law, which then returns a thrust direction and on/off flag.
- Singularities arise in the *propagation* of the classical orbital elements EOMS. The Q-law is not as susceptible to these singularities, although there can be numerical round-off issues which must be addressed by suitable expansions.
- Q-law makes no attempt to exploit the particulars of a given engine (e.g. the small increase in thrust acceleration over time for a constant-thrust, constant-I_{sp} engine).
- Q-law also does not exploit perturbations to the dynamics (but could do so) .

Problem Description

Use: **Classical orbit elements,** $\text{æ} = \boldsymbol{\alpha, e, i, \omega, \Omega}$
True anomaly, $\boldsymbol{\theta}$

<u>Goal:</u>

$$\boxed{\begin{array}{c}\text{æ} \\ \textbf{Current} \\ \textbf{orbit} \\ \textbf{elements}\end{array}} \xrightarrow[\textbf{Thrust}]{\textbf{Apply}} \boxed{\begin{array}{c}\text{æ}_{\boldsymbol{T}} \\ \textbf{Target} \\ \textbf{orbit} \\ \textbf{elements}\end{array}}$$

<u>Must Determine:</u>

- **When to apply thrust**
- **In what direction to apply it**

Solution Method

Lyapunov Feedback Control

1. Determine how "far" it is from $\boldsymbol{\oe}$ **to** $\boldsymbol{\oe_T}$

2. Determine thrust direction that gets us "closer" most quickly

Effectivity Concept

3. Do not thrust now if it is more effective to do so later

Proximity-Quotient (Q) Concept

Q indicates how "close" we are to the target orbit:

$$\boldsymbol{Q} = (1 + \boldsymbol{W_P P}) \sum_{œ} \boldsymbol{W}_{œ}\, \boldsymbol{S}_{œ} \left[\frac{\boldsymbol{d}(œ, œ_{\boldsymbol{T}})}{\dot{œ}_{\mathbf{xx}}}\right]^2$$

where

$$\boldsymbol{d}(œ, œ_{\boldsymbol{T}}) = \begin{cases} œ - œ_{\boldsymbol{T}} & \textbf{for } œ = \boldsymbol{a, e, i} \\ \mathbf{cos}^{-1}\left[\mathbf{cos}(œ - œ_{\boldsymbol{T}})\right] & \textbf{for } œ = \boldsymbol{\omega}, \boldsymbol{\Omega} \end{cases}$$

$$\boldsymbol{S}_{œ} = \begin{cases} \left[\mathbf{1} + \left(\dfrac{\boldsymbol{a} - \boldsymbol{a_T}}{\boldsymbol{m a_T}}\right)^{\boldsymbol{n}}\right]^{\frac{1}{r}} & \textbf{for } œ = \boldsymbol{a} \\ \mathbf{1} & \textbf{for } œ = \boldsymbol{e, i, \omega, \Omega} \end{cases}$$

Q-law Feedback Control

- **Thrust to get "closer" ($Q \downarrow$) as quickly as possible:**

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \sum_{\text{œ}} \frac{\partial Q}{\partial \text{œ}} \dot{\text{œ}}$$

- **Thrust only if we can now get "closer" effectively enough, compared to the optimum location on the osculating orbit to get "closer":**

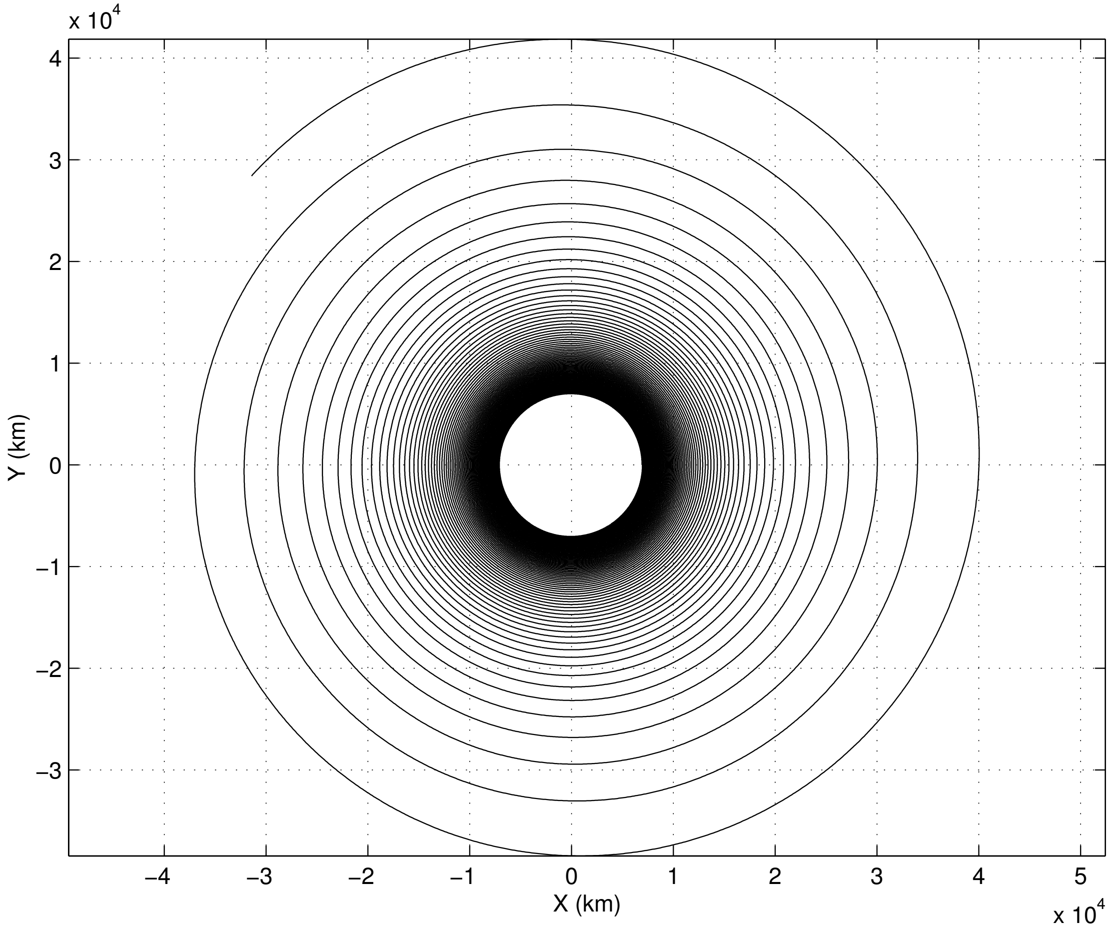
$$\eta = \frac{\dot{Q}_{\mathrm{n}}}{\dot{Q}_{\mathrm{nn}}}$$

Thrust if

$$\eta > \eta_{\mathrm{cut}}$$

Case A: LEO–GEO, coplanar

$\boldsymbol{\eta_{\text{cut}} = 0}$, **Thrust = 1 N**, $\boldsymbol{m_0 = 300}$ **kg**, $\boldsymbol{I_{sp} = 3100}$ **s**

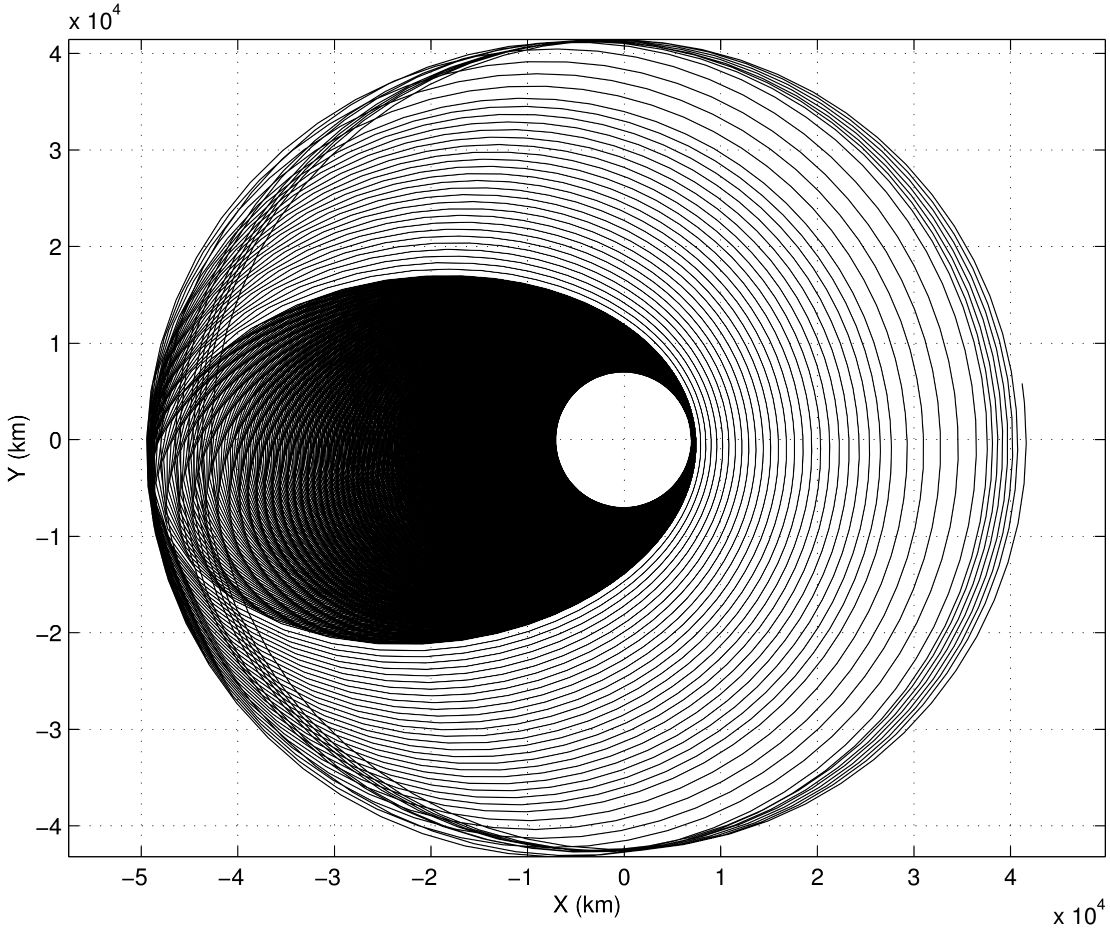


Q-law: TOF = 14.60 days, $\boldsymbol{\Delta V}$ **= 4.53 km/s, 90.38 Revs**

Edelbaum TOF = 14.42 days, $\boldsymbol{\Delta V}$ **= 4.47 km/s,**

Case A: LEO–GEO, coplanar

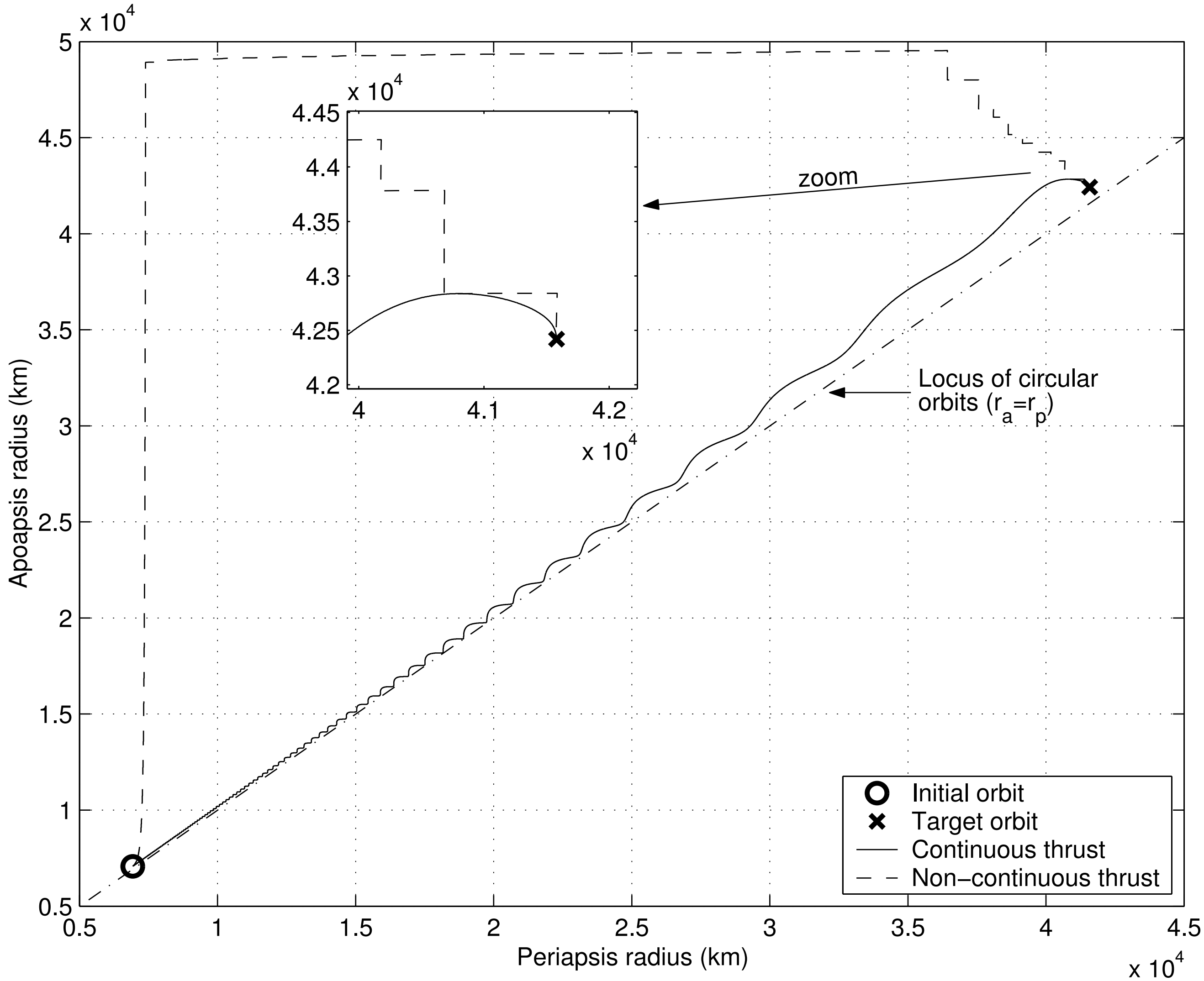
$\boldsymbol{\eta_{\text{cut}} = 0.968}$



Q-law: TOF = 152.4 days, $\boldsymbol{\Delta V}$ = 3.95 km/s, 666.02 Revs
Hohmann TOF = 0.22 days, $\boldsymbol{\Delta V}$ = 3.77 km/s

Case A: LEO–GEO, coplanar

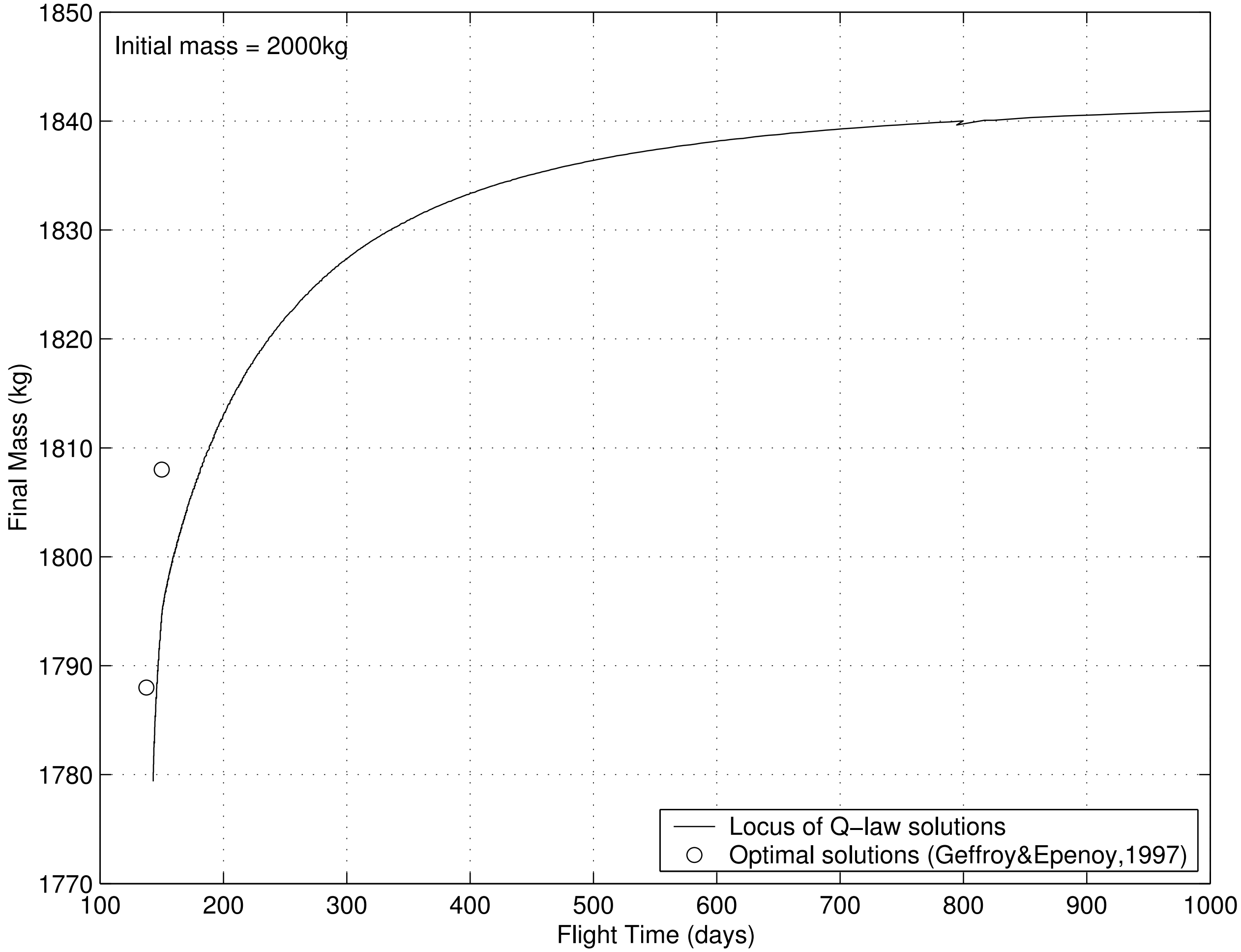
Comparison of paths in r_a-r_p space



Case B: GTO–GEO, $\boldsymbol{\Delta i = 7°}$

Thrust = 0.35 N, $\boldsymbol{m_0}$ = 2000 kg, $\boldsymbol{I_{sp}}$ = 2000 s

Trade-off between final mass and flight time



Other Lyapunov functions for orbit transfers

- Ilgen, 1993 — earliest found in literature

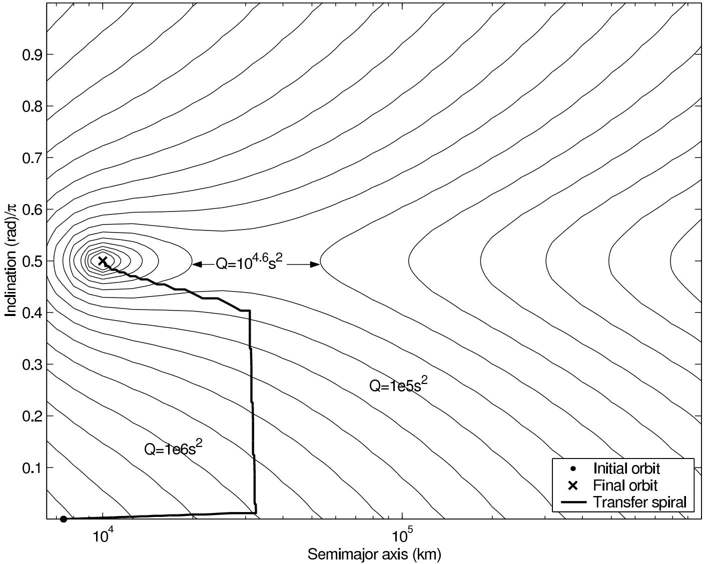
$$
\begin{aligned}
V &= W_a(a - a_T)^2 + W_e(e - e_T)^2 + W_i(i - i_T)^2 \\
&\quad + W_\omega(\omega - \omega_T)^2 + W_\Omega(\Omega - \Omega_T)^2
\end{aligned}
$$

- Chang, Chichka, Marsden, 2002

$$
V = W_h \left| \vec{h} - \vec{h_T} \right| + W_e \left| \vec{e} - \vec{e_T} \right|
$$

- Various others since 2002: Naasz, Joseph, Bombrun,
- Coupling between elements typically ignored or underutilised
- Good survey of methods: Noble Hatten, MS thesis, 2012, UT Austin

Coupled elements, e.g. a, i



Q-law Feedback Control

- Thrust to get "closer" ($Q \downarrow$) as quickly as possible:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \sum_{œ} \frac{\partial Q}{\partial œ} \dot{œ}$$

- Thrust only if we can now get "closer" effectively enough, compared to the optimum location on the osculating orbit to get "closer":

$$\eta_{\mathrm{a}} = \frac{\dot{Q}_{\mathrm{n}}}{\dot{Q}_{\mathrm{nn}}} \qquad \eta_{\mathrm{r}} = \frac{\dot{Q}_{\mathrm{n}} - \dot{Q}_{\mathrm{nx}}}{\dot{Q}_{\mathrm{nn}} - \dot{Q}_{\mathrm{nx}}}$$

Thrust if

$$\eta_{\mathrm{a}} > \eta_{\mathrm{a\,cut}} \qquad \text{and/or} \qquad \eta_{\mathrm{r}} > \eta_{\mathrm{r\,cut}}$$

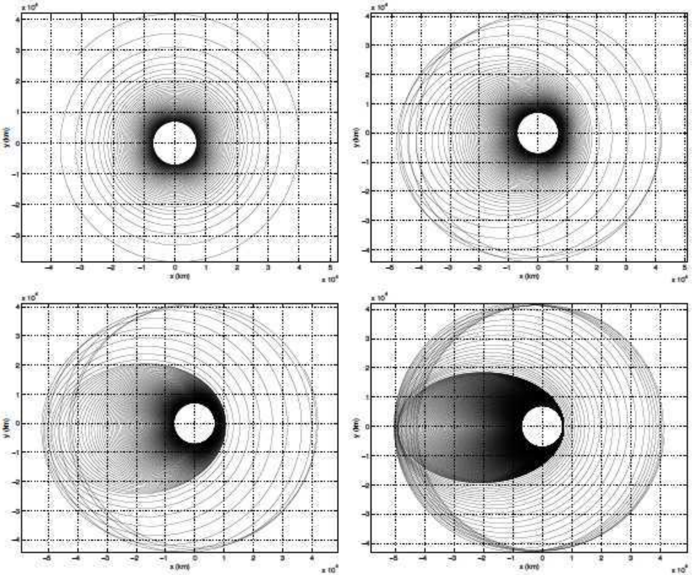
Case A: LEO–GEO, Coplanar

$\eta_{r\,cut} =$ 0

0% Coast

$\eta_{r\,cut} =$ 0.167

42% Coast



$\eta_{r\,cut} =$ 0.435

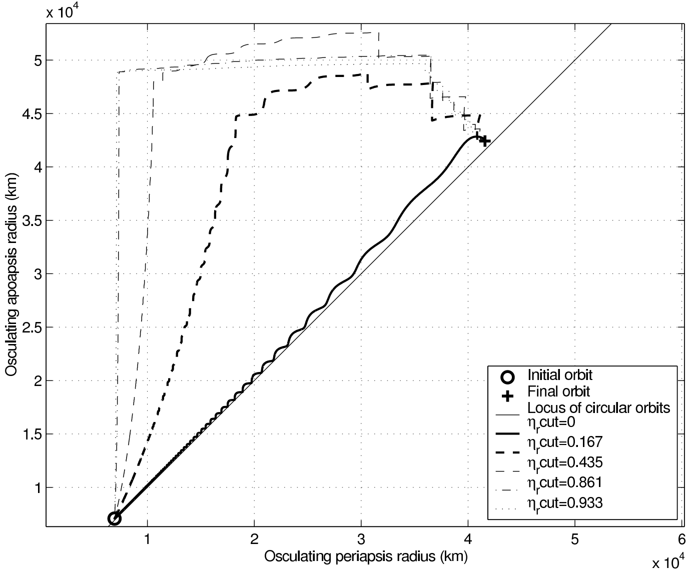
62% Coast

$\eta_{r\,cut} =$ 0.861

87% Coast

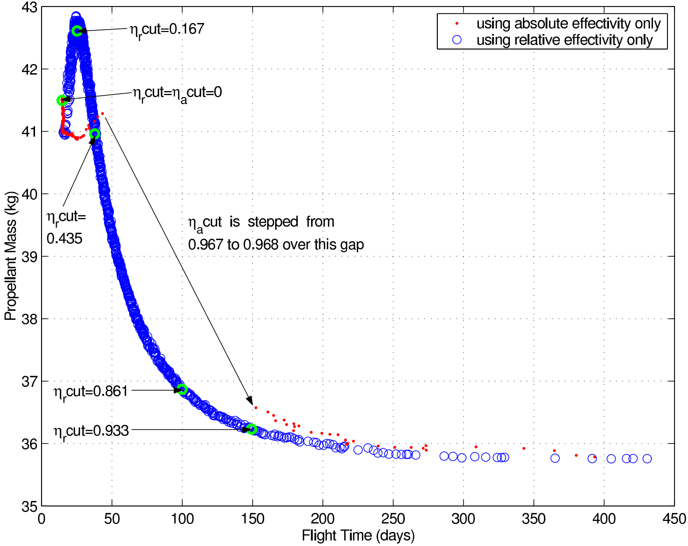
Case A: LEO–GEO, Coplanar

Comparison of paths in r_a-r_p space



Case A: LEO–GEO, Coplanar

Trade-off between propellant mass and flight time

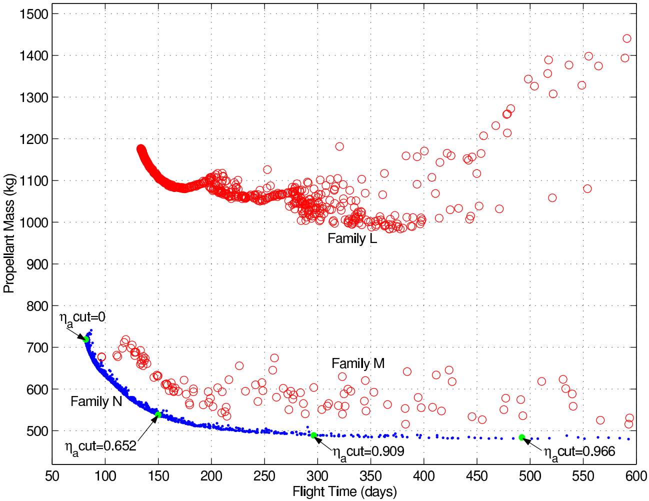


Case E: GTO to Retrograde, Molniya-Type

	a	e	i	ω	Ω
Init.	24506 km	0.725	0.06°	0°	0°
Targ.	26500 km	0.700	116°	270°	180°

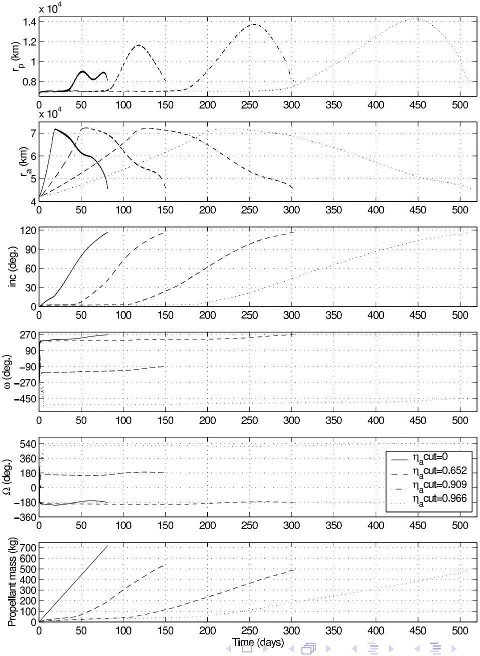
Thrust= 2 N
$m_0 = 2000$ kg
$I_{sp} = 2000$ s

Trade-off between propellant mass and flight time



Case E: GTO to Retrograde, Molniya-Type

Orbit element history



A further word on propagation and control — chatter

- Propagation can be speeded up 100-fold using averaging
- All Lyapunov functions that are a sum of terms are susceptible to thrust-direction chatter
- Chatter will often drive variable-step-size propagators to an infinitesimal step size
- Chatter can be mitigated by temporarily re-setting the effectivity cut-offs to higher values
- Chatter control is possible through other means too, e.g. temporarily ignoring an element's target value, switching elements, re-weighting
- Chatter can also be avoided by phasing — avoiding passage through regions where the conflicting addends (e.g. a and e terms) are equally weighted. This can be done by changing the constant weights or the effectivity cutoffs.

$$V = W_a(a - a_T)^2 + W_e(e - e_T)^2 + \ldots$$

GA-Qlaw I

- Optimise the Pareto front in propellant mass and flight time by varying the Q-law weights and other parameters using a genetic algorithm
- Formally stated:

$$
\begin{aligned}
\text{minimize } \mathbf{y} &= \{t_f(\mathbf{x}), m_p(\mathbf{x})\} \in \mathbf{Y}, \\
\text{where } \mathbf{x} &= \{W_a, W_e, W_i, W_\omega, W_\Omega, W_P, \\
&\quad m, n, r, r_{\text{pmin}}, k, \eta_{\text{cut}}, \theta_i\} \in \mathbf{X}.
\end{aligned}
$$

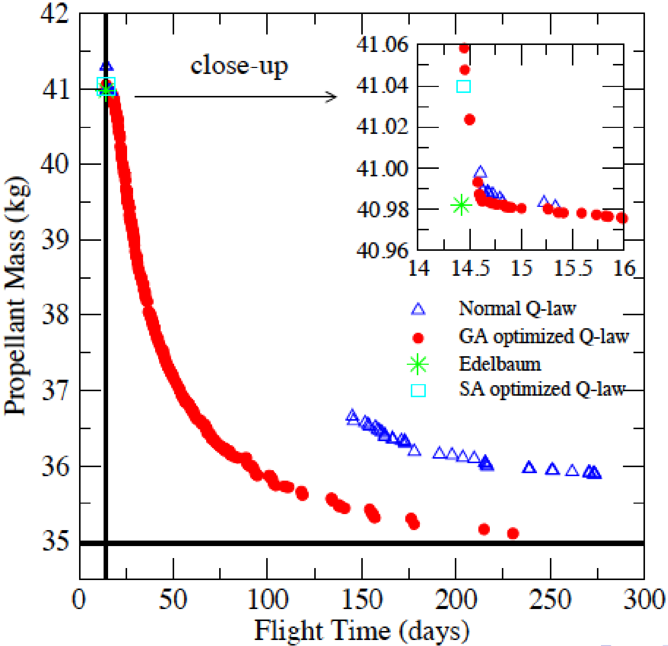
- Well suited for parallel computation on a cluster
- Use non-dominated sorting genetic algorithm, superior to other multi-objective evolutionary algorithms such as Vector-Evaluated GA, niched Pareto GA, multi-objective GA.

(From Lee *et al.*)

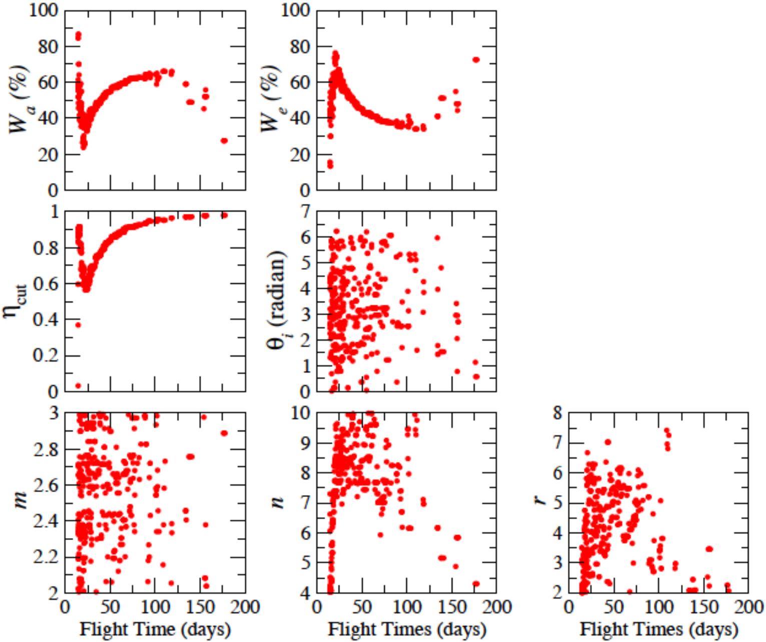
GA-Qlaw II

- Parameters represented as real valued genes
- Fitness of an individual (i.e. a parameter set) based on Pareto rank
- An individual's fitness determines the probability of being selected as a parent
- Crossover and mutation probabilities must be tuned
- Population retention rate must also be tuned
- Can also use mandatory selection for a handful of high-fitness cases, i.e. use some 'niche-ing'
- Parents are selected by tournament — e.g. pick four parents at random, keep the best two (two parents give one off-spring)

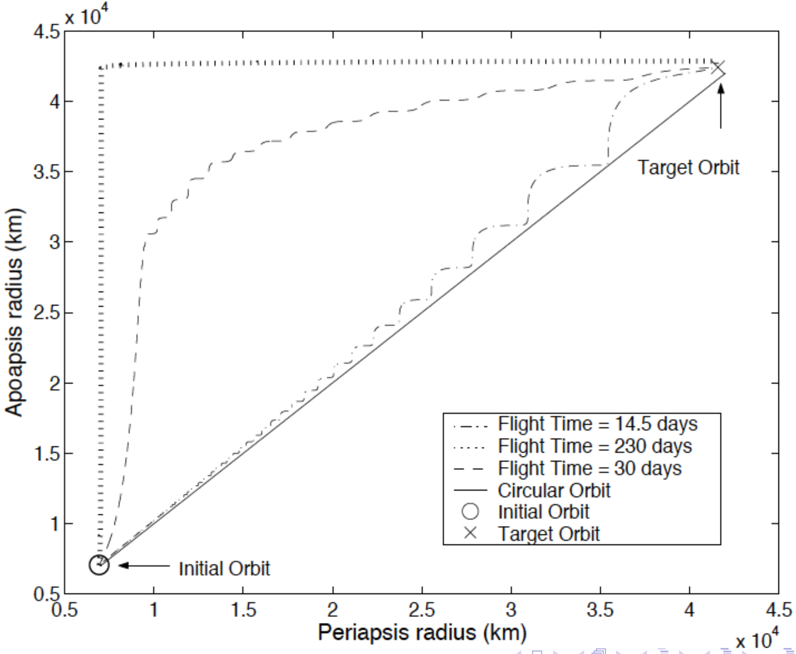
GA-Qlaw Case A, I



GA-Qlaw Case A, II



GA-Qlaw Case A, III



Methods considered

Optimal Lyapunov Feedback Control:

- Q-Law
- GA-QLaw

Indirect Optimisation Methods:

- Ztool (with averaging)
- T_3D (with and without averaging)
- MIPELEC (with averaging)

Direct Optimisation Method:

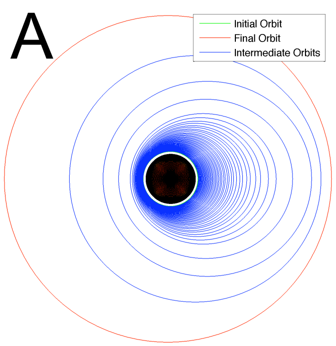
- OPTIFOR

TEST-CASE ORBIT TRANSFERS

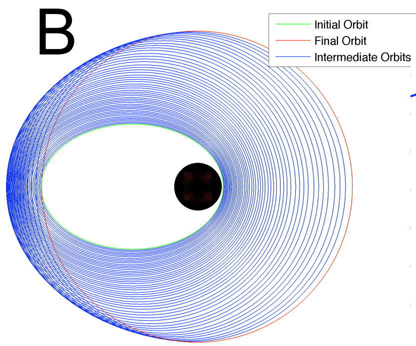
Point-mass central body, no perturbations.

Case	Orbit	*a*, km	*e*	*i*, deg.	*ω*, deg	*Ω*, deg	Thrust, N	Initial Mass, kg	Specific Impulse, s	Central Body
A	Init.	7000	0.01	0.05	0		1	300	3100	Earth
	Targ.	42000	0.01	free	free	free				
B	Init.	24505.9	0.725	7.05	0	0	0.350	2000	2000	Earth
	Targ.	42165	0.001	0.05	free	free				
C	Init.	9222.7	0.2	0.573	0	0	9.3	300	3100	Earth
	Targ.	30000	0.7	free	free	free				
D	Init.	944.64	0.015	90.06	156.90	-24.60	0.045	950	3044.9	Vesta
	Targ.	401.72	0.012	90.01	free	-40.73				
E	Init.	24505.9	0.725	0.06	0	0	2	2000	2000	Earth
	Targ.	26500	0.7	116	270	180				

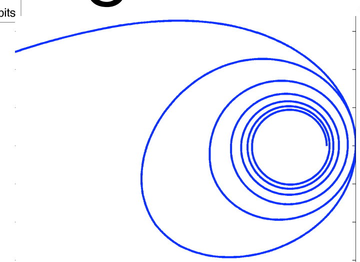
A



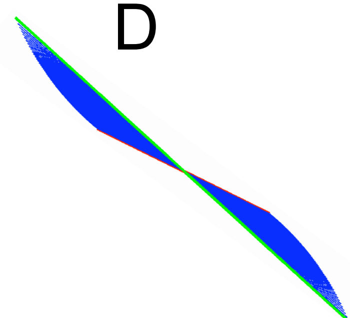
B



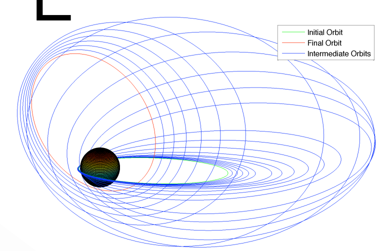
C



D



E



ZTool – Indirect Averaging Method

- Equinoctial elements used for the dynamics
- Thrust expressed in equinoctial frame or RTN frame
- Optimal Control Problem:
  - α is useful for continuation

$$J = \alpha m^f + (1-\alpha)\int_{t_0}^{t_f} -\frac{T^2}{2}dt$$

$$H = -\frac{dJ}{dt} + \sum_{i=1}^{6} \lambda_i \frac{dx_i}{dt}$$

- Time is the independent variable

$$\frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x_i}$$

subject to boundary conditions

- Averaging is over eccentric longitude: $d\bar{x}_i / dt = \frac{\varepsilon}{2\pi}\int_{\bar{F}-\pi}^{\bar{F}+\pi} g_{x_i}\left(\bar{x}_{1\ldots5,7}, \bar{\lambda}_{1\ldots5,7}, \bar{F}\right)\frac{d\bar{L}}{d\bar{F}}\, d\bar{F} + O(\varepsilon^2) \qquad i=$
- Gaussian averaging for the thrust
- Transformation between mean and osculating elements
- Perturbations are included in the formulation but not studied here

T_3D and MIPELEC

Main differences with Ztool algorithm:

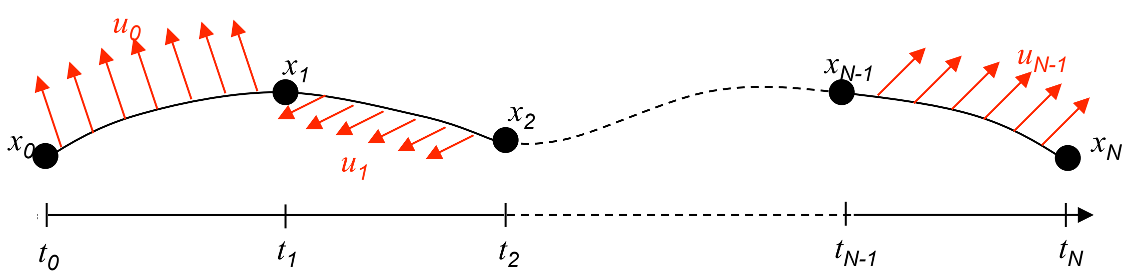
- T_3D
  - averages over time
  - more complex cost function
  - smoothing for bang-bang thrust and shadowing
  - auto-initialisation method for co-states based on analytic theories
- MIPELEC
  - minimum time problem only
  - averaging over true longitude

T_3D – see companion paper by Dargent at this conference

MIPELEC – source available from CNES

Direct Method – OPTIFOR

- Transform the continuous optimal control problem into a discrete nonlinear programming problem
  - Easy to implement and more robust with respect to starting estimates
  - But large number of variables (slow)
- Discretisation with segments of constant thrust



Time or true longitude discretization

Equinoctial elements

Thrust expressed in the radial-transverse-normal frame

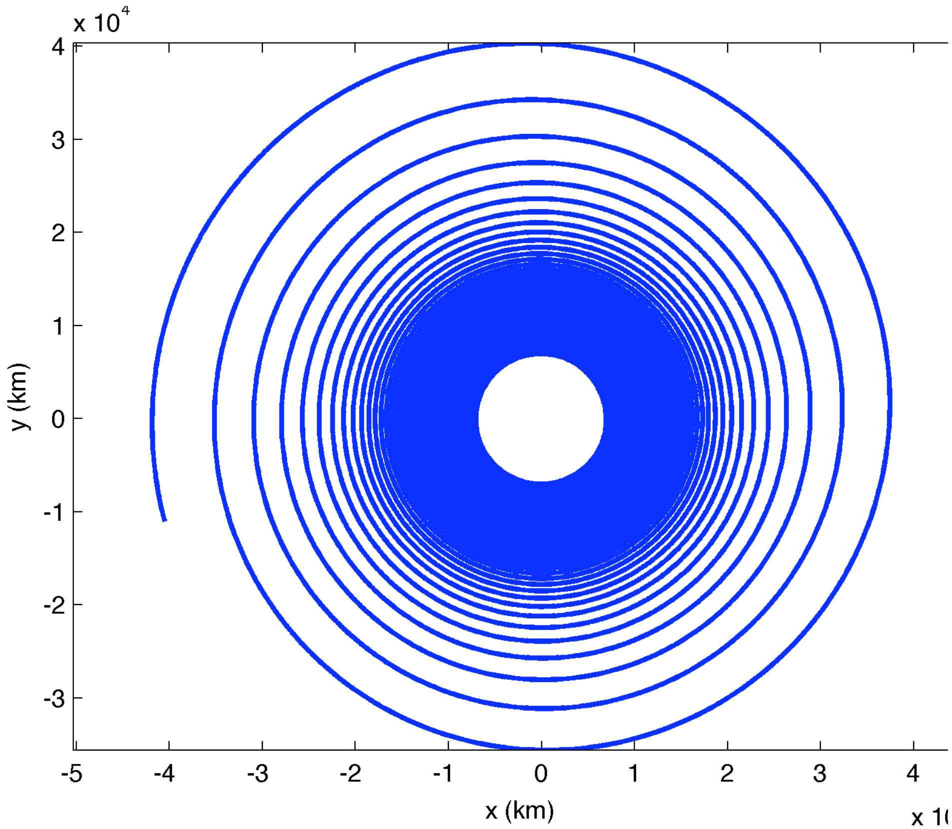
- Problem Formulation
  - Solved using SNOPT (sparse NLP solver), within the OPTIFOR optimization

$$\min_{u_0,\dots,u_{N-1}} J = \sum_{k=0}^{N-1} L_k(x_k, u_k) + \phi(x_N)$$
Final mass or final time

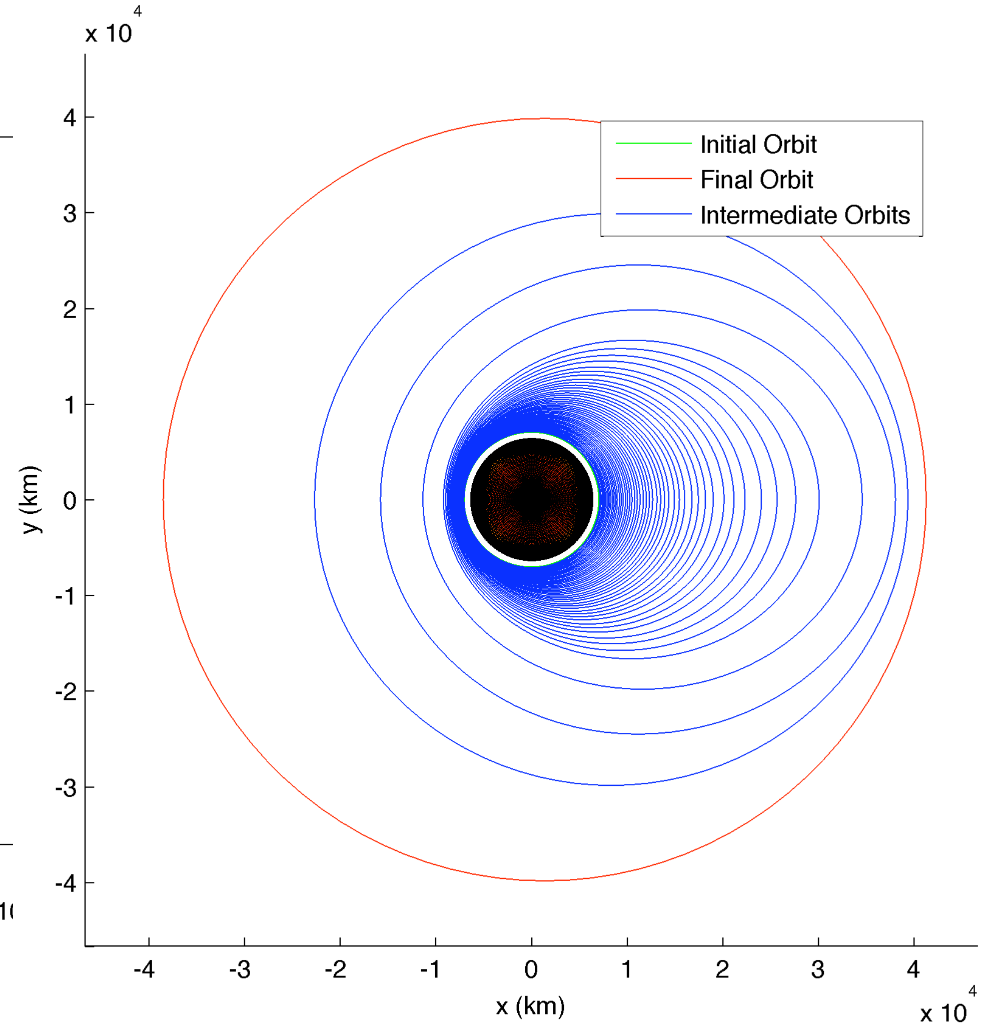
subject to
$$\begin{cases} x_{k+1} = F_k(x_k, u_k) \\ g_k(x_k, u_k) = 0 \\ h(x_N) = 0 \end{cases}$$

- $x_{k+1} = F_k(x_k, u_k)$: Numerical integration of equations of motion
- $g_k(x_k, u_k) = 0$: Maximum thrust and minimum periapsis
- $h(x_N) = 0$: Targeted orbital elements

TRAJECTORIES: CASE A (LEO TO GEO)

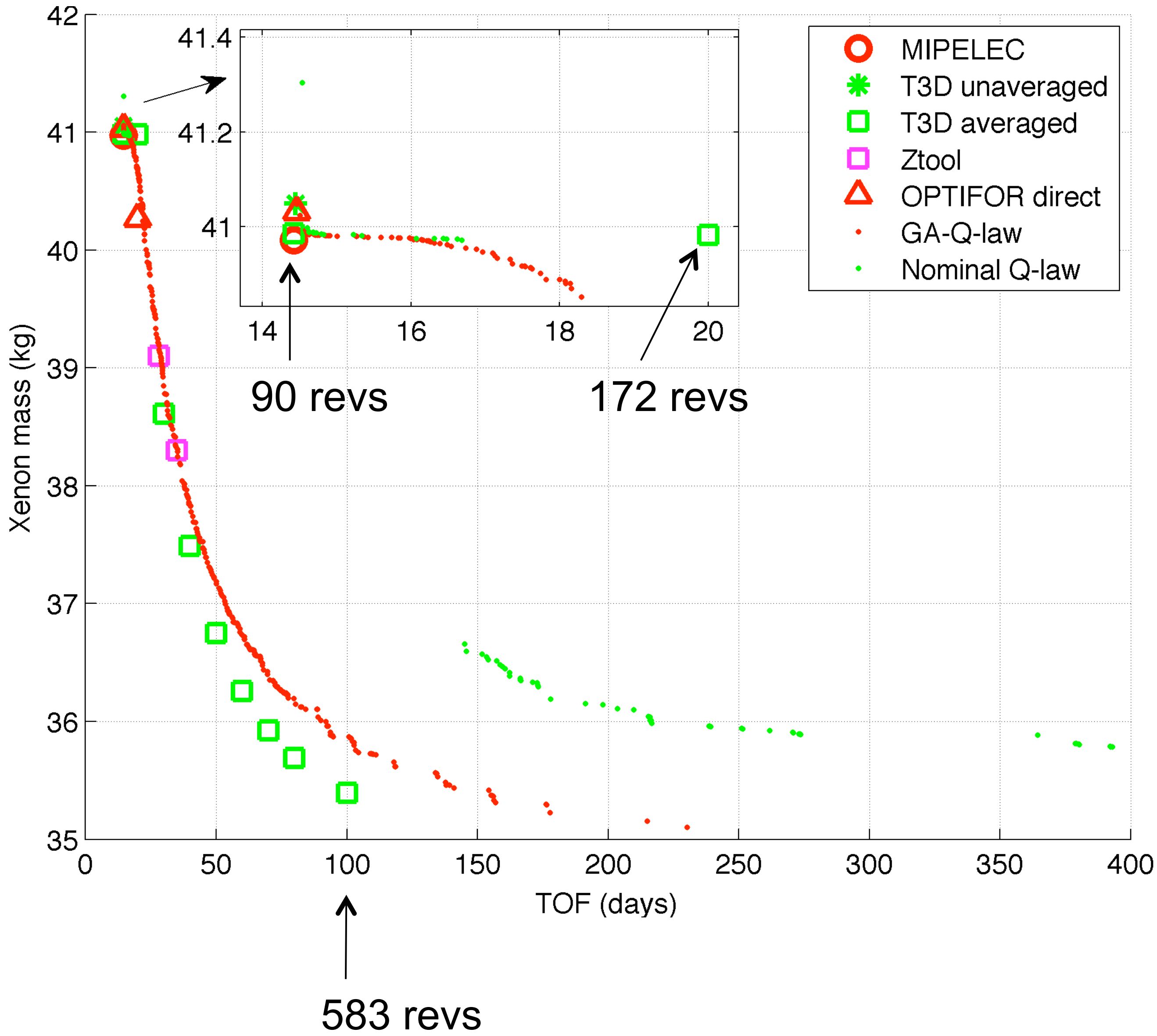


OPTIFOR, Min Time,
14.4 days, 90 revs



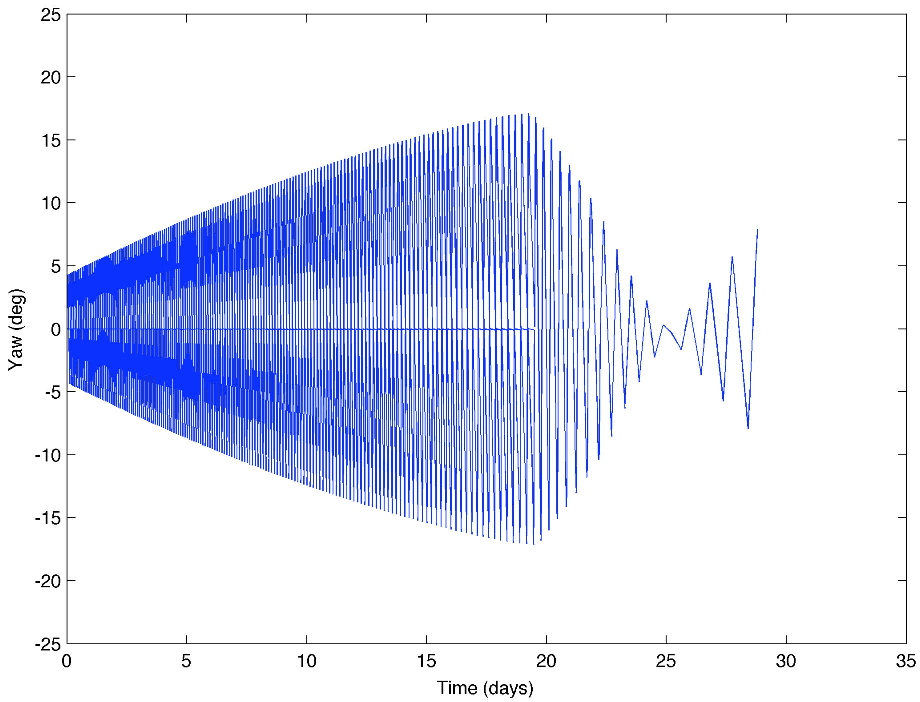
Ztool, Min Fuel
35 days, 221 revs
not all intermediate orbits plotted

Mass Performance: Case A

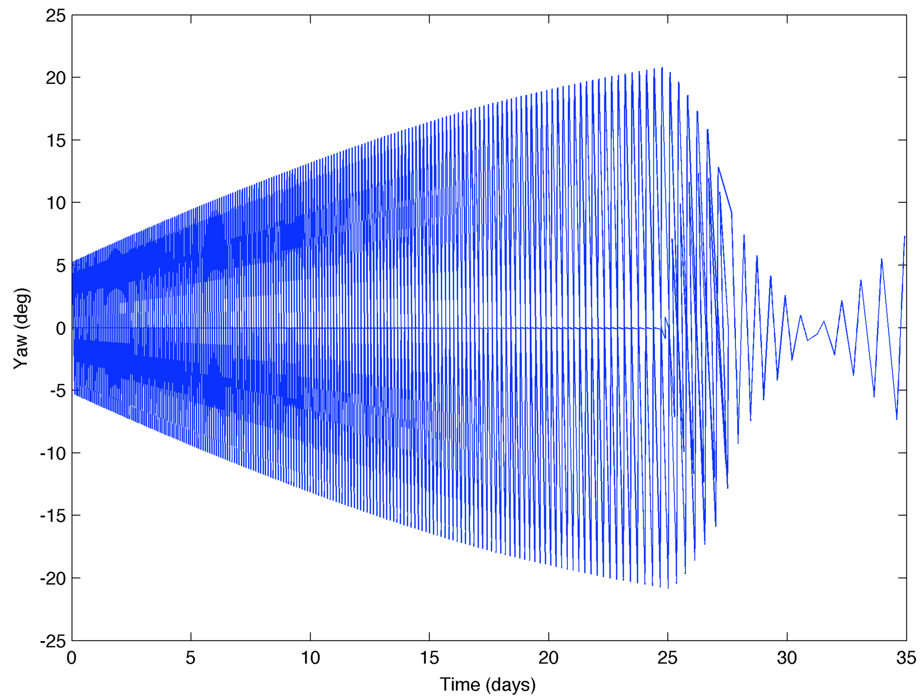


- Good agreement between the tools in fuel mass
- Q-law ~0-5 % from optimal fuel mass
- GA-Qlaw ~0-2 %

Thrust Direction: Case A



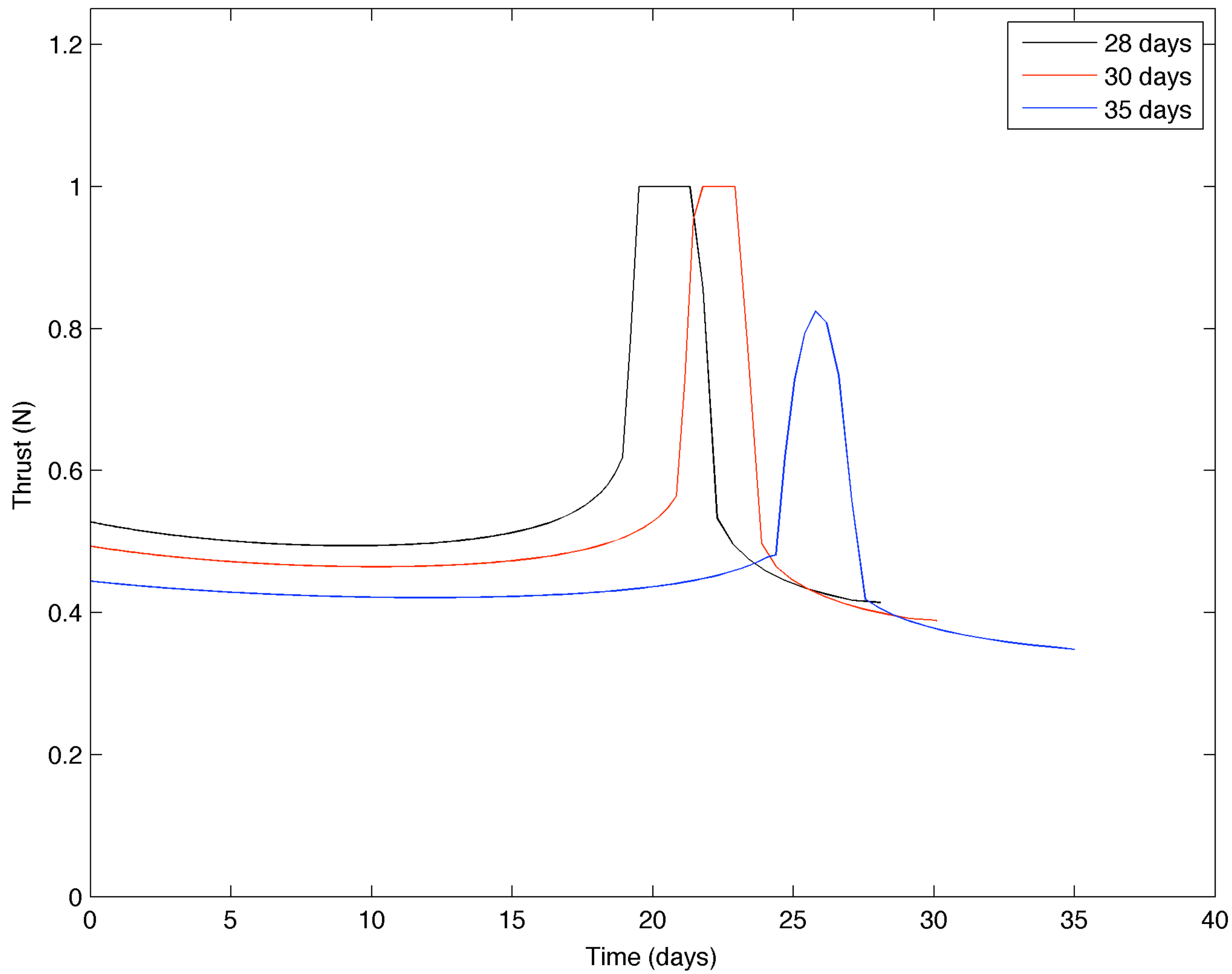
Ztool, 28-day TOF



Ztool, 35-day TOF

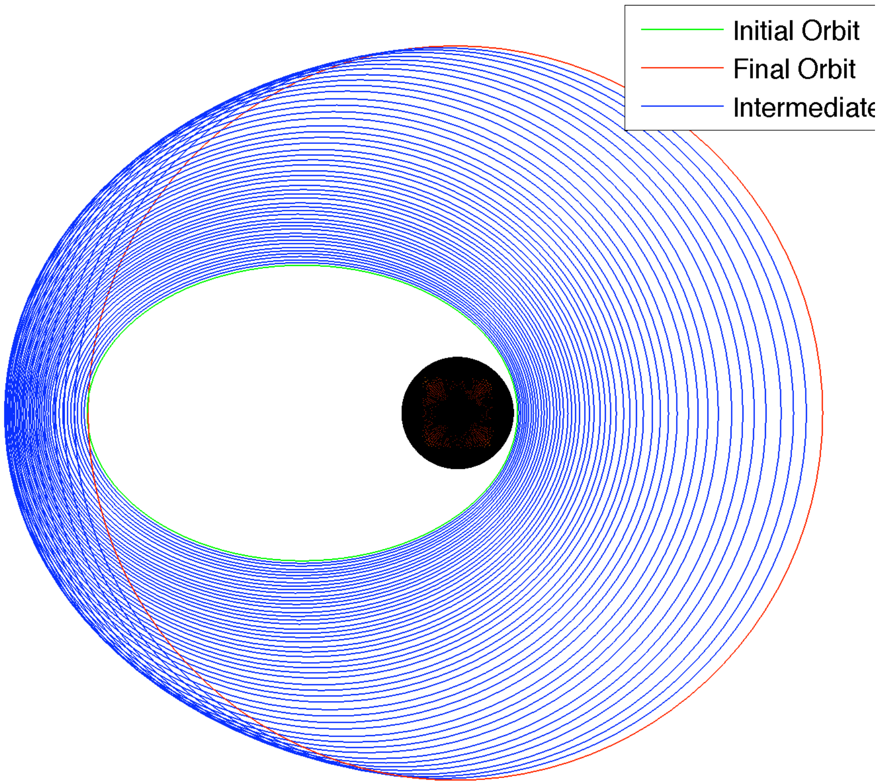
Yaw angle of thrust (off of circumferential direction) varies more for longer transfers – eccentricity effect

Average Thrust Magnitude: Case A

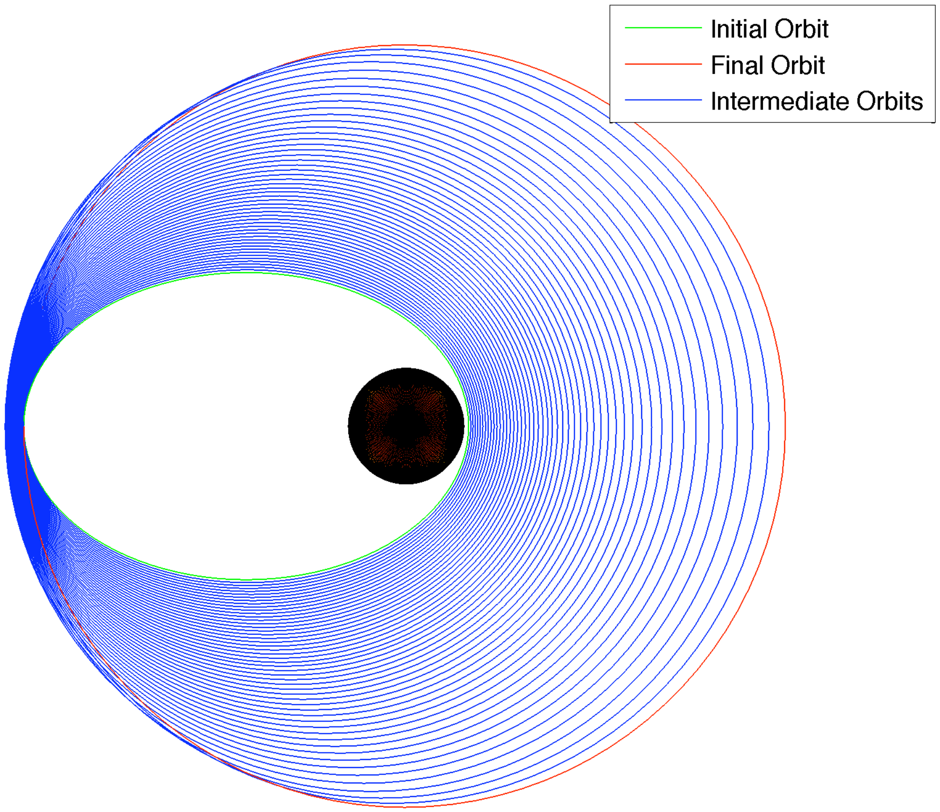


For intermediate flight times, there are several revs in the latter part of the transfer where thrust is applied continuously (average thrust = max thrust = 1N)

TRAJECTORIES: CASE B (GTO TO GEO)

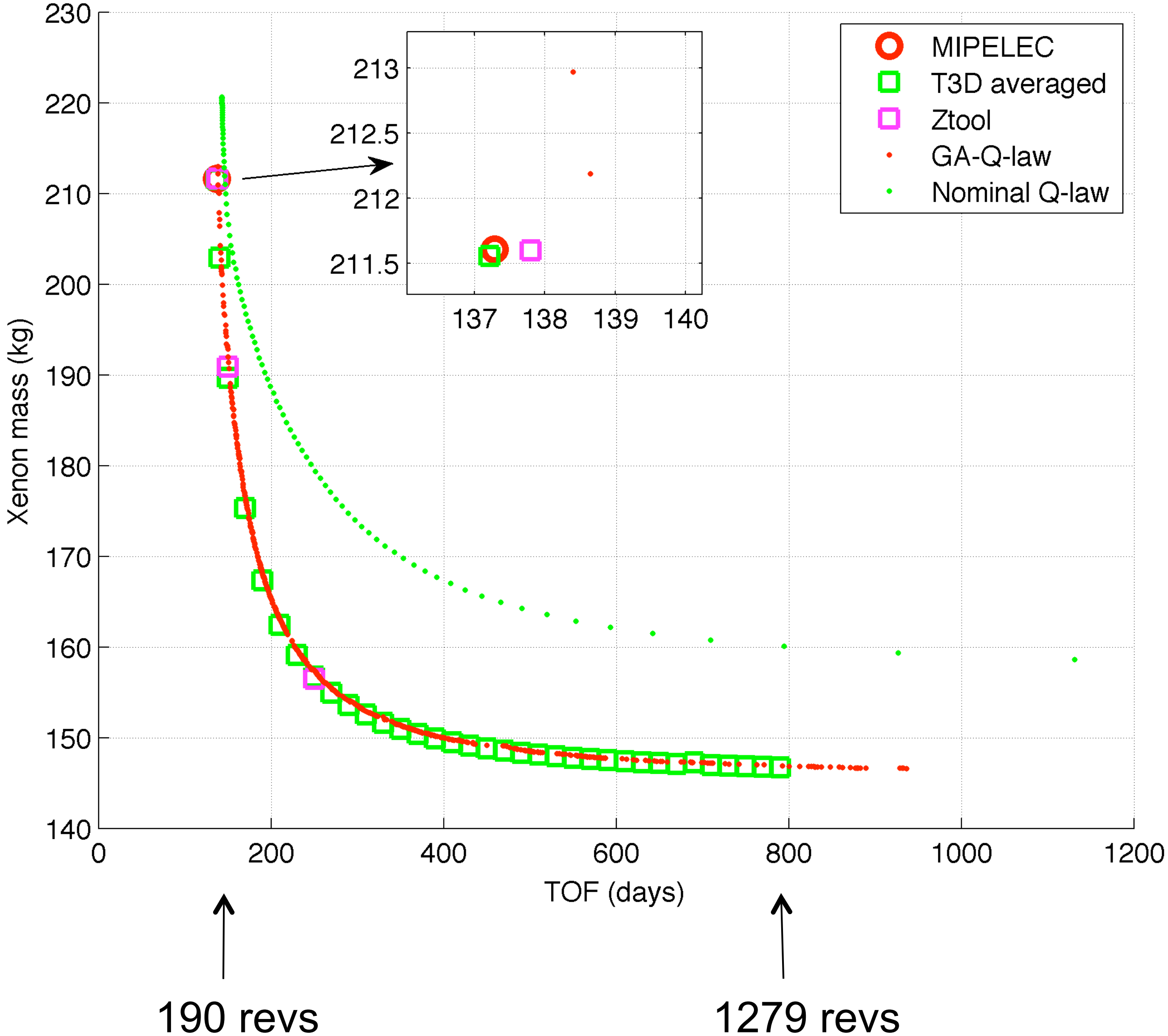


Ztool, 150-day TOF, 217 revs



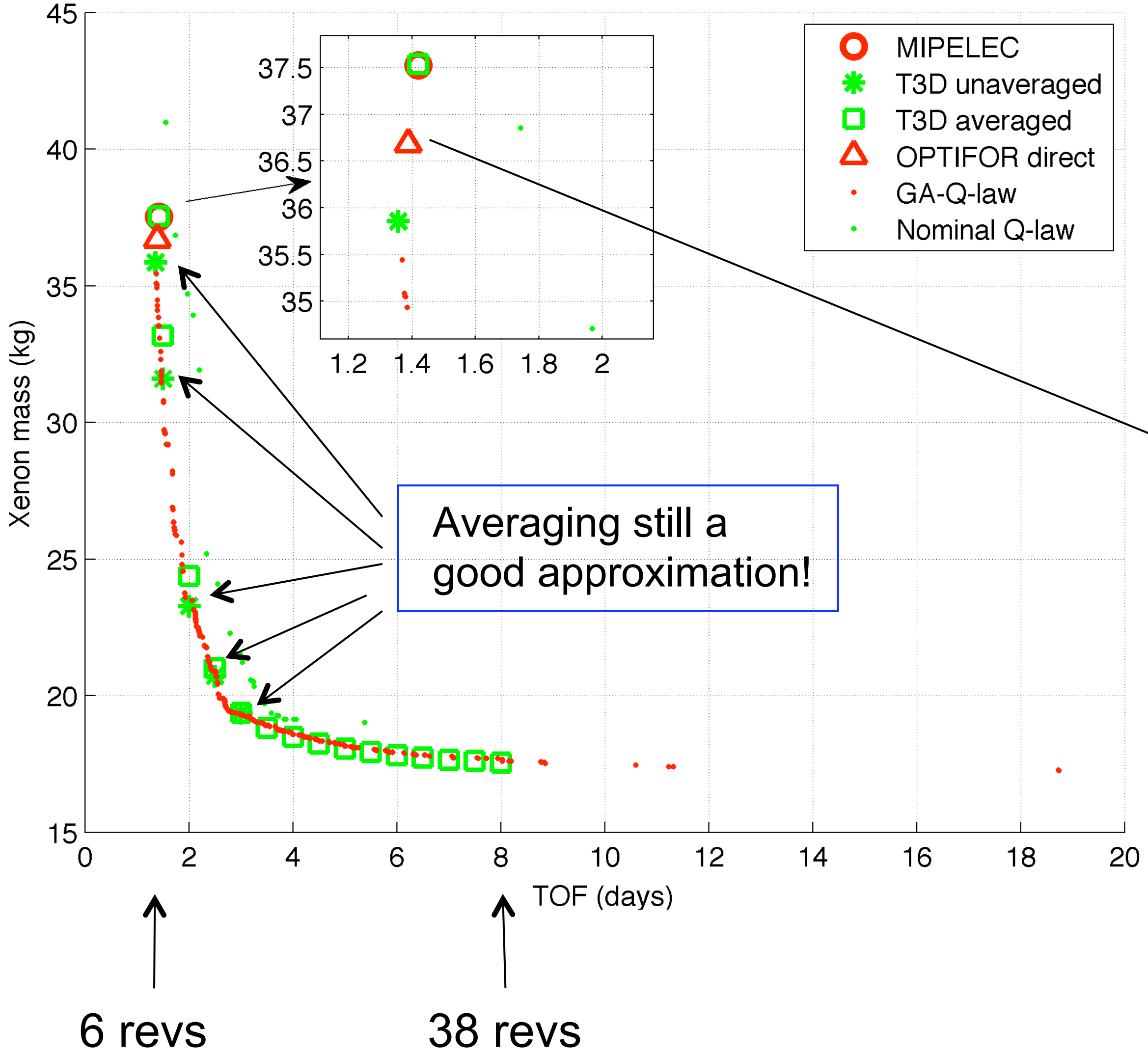
Ztool, 250-day TOF, 398 revs

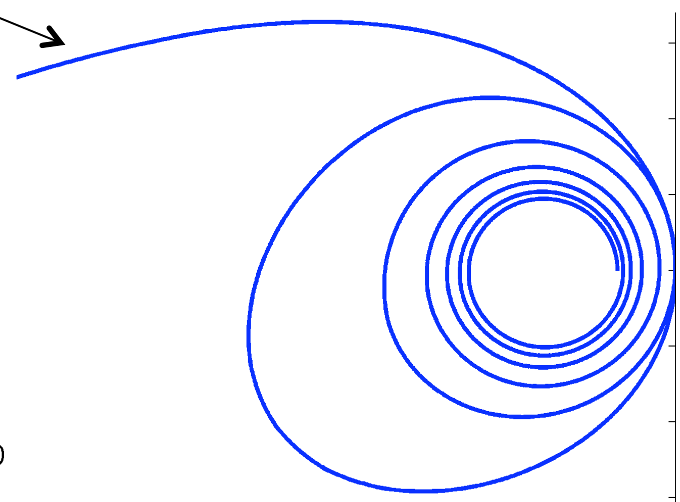
Mass Performance: Case B



- Good agreement between the tools in fuel mass
- Q-law ~5-10 % from optimal fuel mass
- GA-Qlaw ~0-1 %

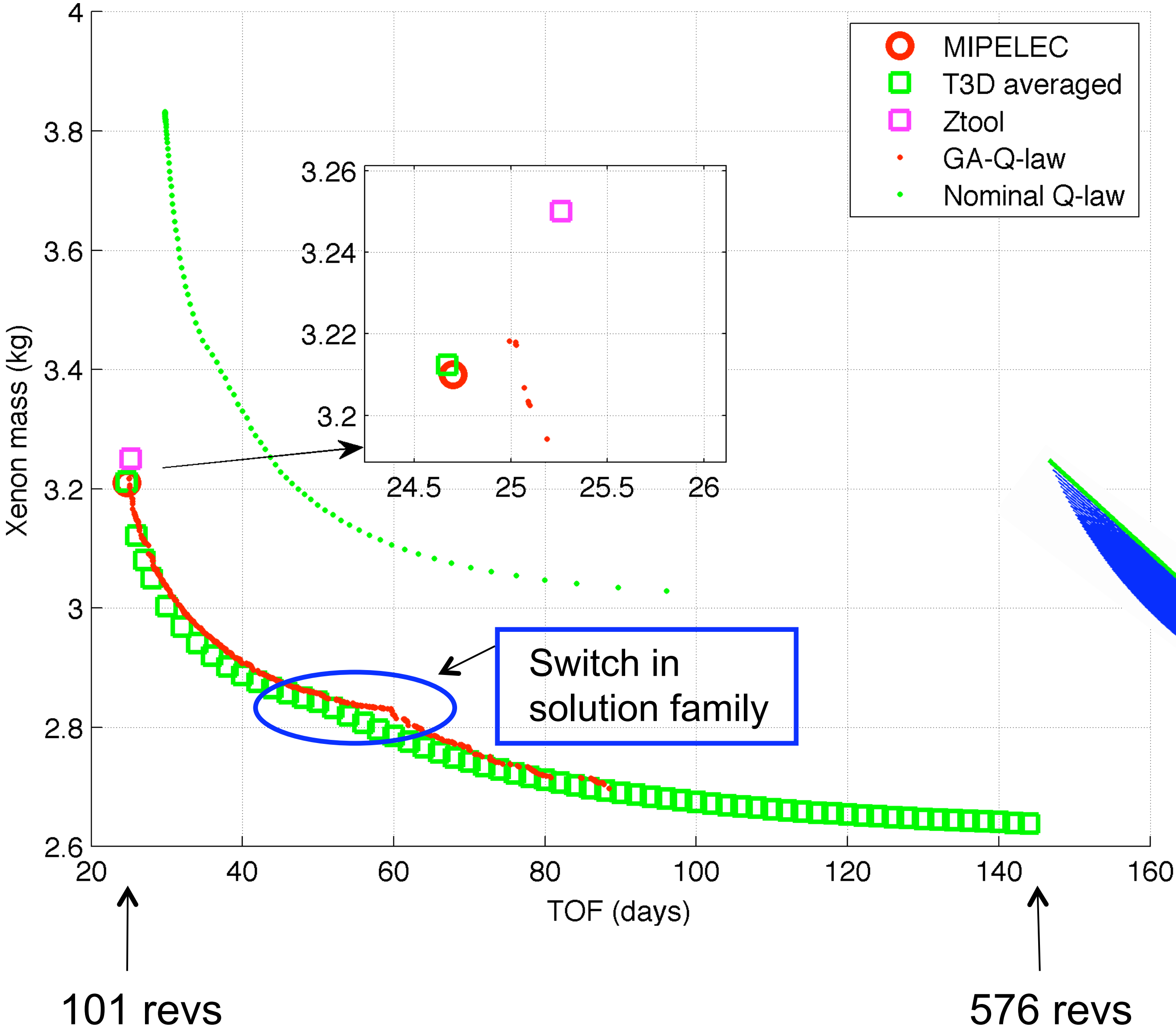
Case C – "High" Thrust



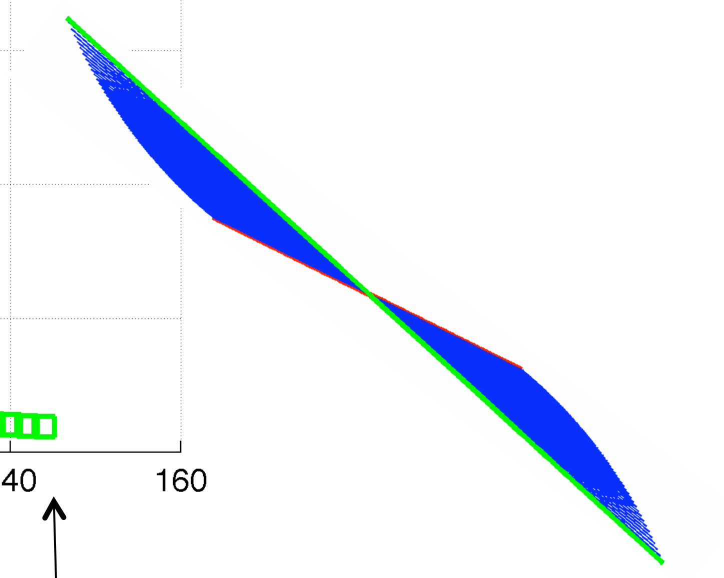


- Good agreement between the tools in fuel mass

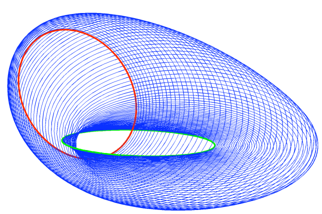
Mass Performance: Case D (Vesta, 4 elements)



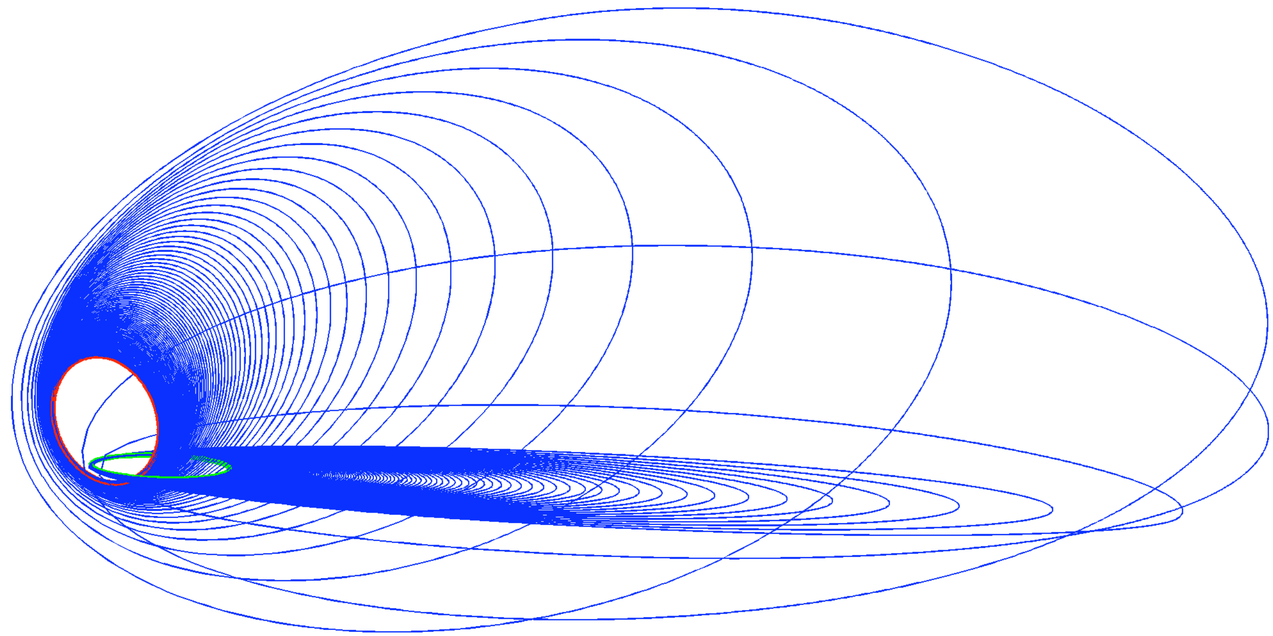
- Good agreement between the optimizers
- Q-law ~15-20 % from optimal fuel mass
- GA-Qlaw ~0-1 %



Trajectories: Case E (5 elements)

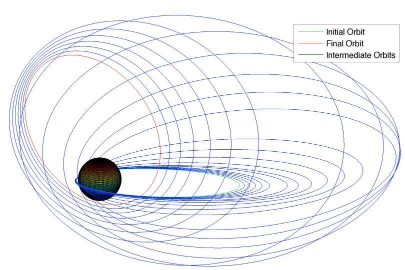


T_3D, Min TOF:
58.5 days, 82 revs



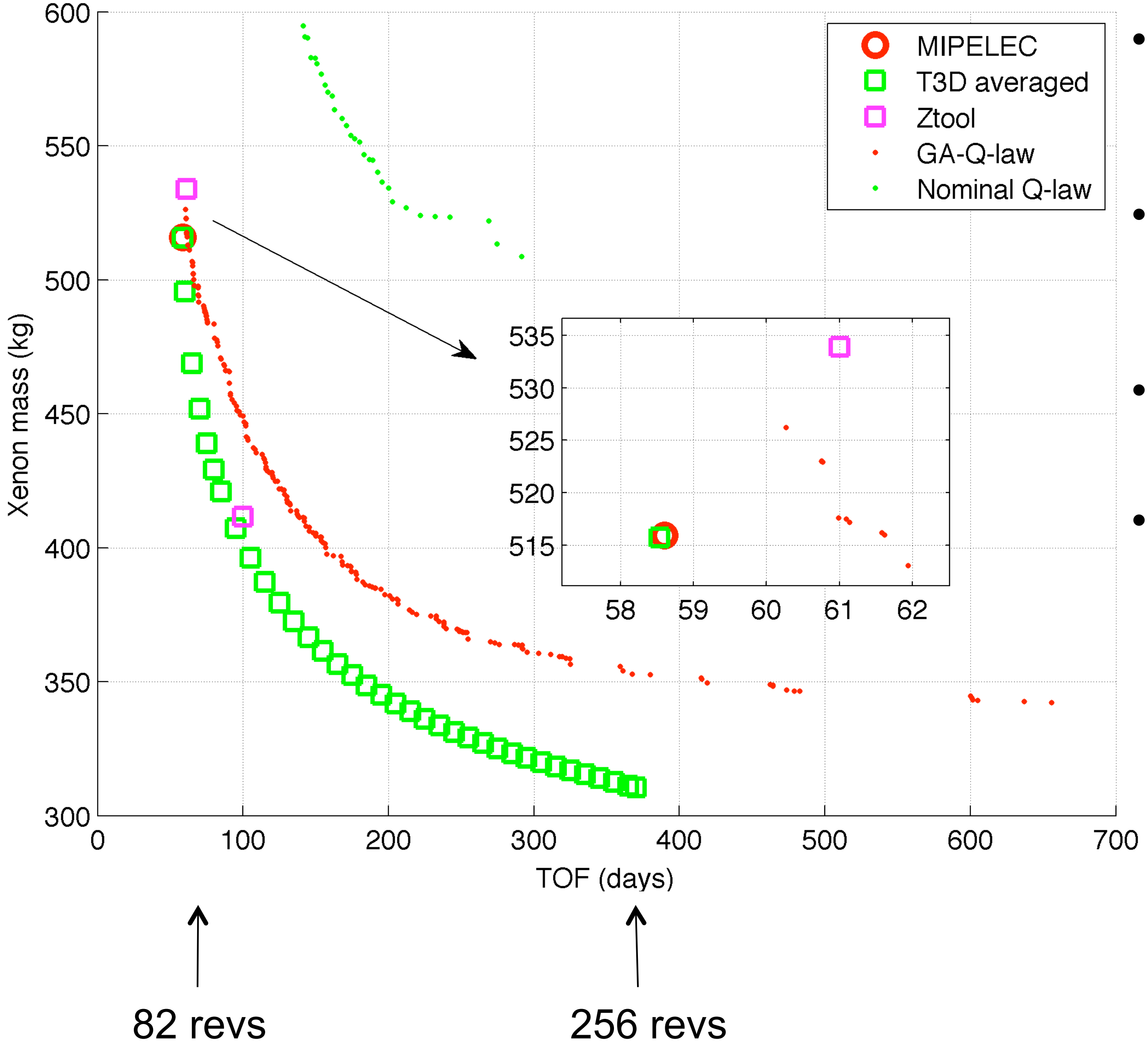
T_3D, Min Fuel:
360 days, 254 revs

same scale



Ztool, Min Fuel:
100 days, 113 revs

Mass Performance: Case E (5 elements)



- Good agreement between the optimizers
- Q-law greatly overestimates fuel mass
- GA-Qlaw ~2-13 %
- CPU run times for generating mass performance curve:
  - Q-law: 10s of seconds
  - T_3D: 10s of minutes
  - GAQLaw: 10s of hours