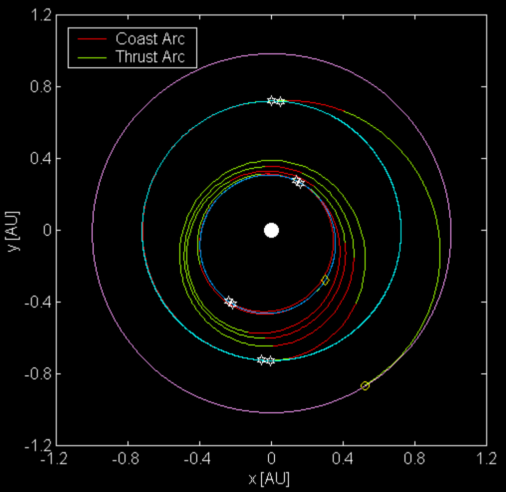

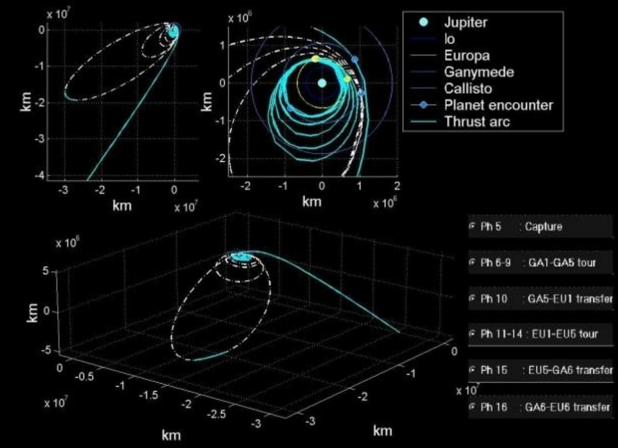

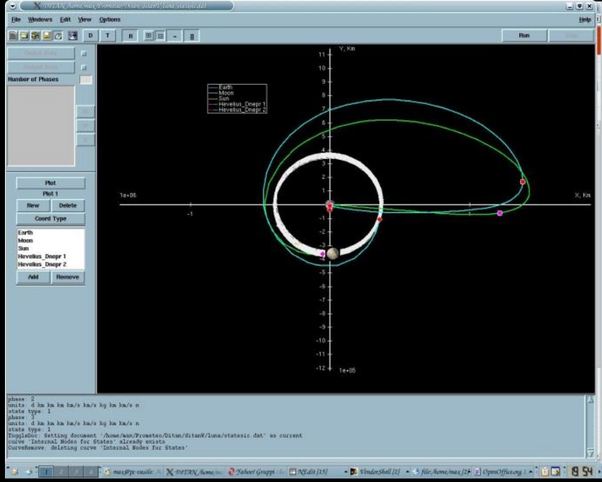

JAXA

FINITE ELEMENTS IN TIME and GAUSS PSEUDO-SPECTRAL METHODS

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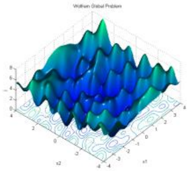
Direct Transcription with FET

Direct Transcription with GPSM



Basics of Nonlinear Programming

Some Examples

Finite Perturbative Elements

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OPTIMAL CONTROL

Optimal Control Problem

- Minimise:

$$J = \varphi\left(\mathbf{x}(t_0), \mathbf{x}(t_f), t_0, t_f\right) + \int_{t_0}^{t_f} L(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

Subject to a set of differential constraints

$$\dot{x}_i(t) = F_i(\mathbf{x}(t), \mathbf{u}(t), t), \quad i = 1, \ldots, n,$$
$$t_0 \leq t \leq t_f$$

Boundary constraints

$$\psi_i(\mathbf{x}(t_0), \mathbf{x}(t_f), t_f) \geq 0, \quad i = 1, \ldots, k \leq 2n$$

And algebraic constraints

$$G_i(\mathbf{x}(t), \mathbf{u}(t), t) \geq 0, \quad i = 1, \ldots, m, \quad t_0 \leq t \leq t_f$$

Example of Optimal Control Problem

- Minimise:

$$J = \varphi\left(\mathbf{x}(t_0), \mathbf{x}(t_f), t_0, t_f\right) = v(t_f)$$

Subject to a set of differential constraints

$$\dot{x} = v$$
$$\dot{v} = \frac{T - D}{m} - g;$$
$$\dot{m} = -\frac{T}{g_0 I_{sp}}$$

Boundary constraints

$$\psi_1(\mathbf{x}(t_0), \mathbf{x}(t_f), t_f) = h(t_0) = 0$$
$$\psi_2(\mathbf{x}(t_0), \mathbf{x}(t_f), t_f) = v(t_0) = 0$$
$$\psi_3(\mathbf{x}(t_0), \mathbf{x}(t_f), t_f) = m(t_0) = m_0$$
$$\psi_4(\mathbf{x}(t_0), \mathbf{x}(t_f), t_f) = h(t_f) = h_f$$

And algebraic constraints

$$G_1 = T$$
$$G_2 = T_{\max} - T$$

Pontryagin Maximum Principle

- Consider the optimal control problem:

$$\min_{\mathbf{u}\in U} J(\mathbf{u},t_f) = \varphi\left(\mathbf{x}(t_0),\mathbf{x}(t_f),t_0,t_f\right) + \int_{t_0}^{t_f} L(\mathbf{x}(t),\mathbf{u}(t),t)dt$$

$$subject\ to:$$

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t,\mathbf{x}(t),\mathbf{u}(t))$$

$$\boldsymbol{\psi}(\mathbf{x}(t_0),\mathbf{x}(t_f)) \geq 0$$

If $(\mathbf{u}^*,t_f^*)\in \hat{C}[t_0,T]^q \times [t_0,T)$ is optimal, with response $\mathbf{x}^* \in \hat{C}^1[t_0,T]^n$, being $\hat{C}$ and $\hat{C}^1$ the spaces of piecewise continuous and piecewise continuous and differentiable functions respectively. Then, there exist $\left(\lambda_0^*,\boldsymbol{\lambda}^*\right)\in \hat{C}^1\left[t_0,T\right]^{n+1}$ and $\mathbf{v}^* \in \Re^{n_b}$ such that: $\left[\lambda_0^*(t),\boldsymbol{\lambda}^*(t)\right] \neq [0,0], t_0 \leq t \leq t_f^*$

$$\dot{\lambda}_0(t) = 0, \quad \dot{\boldsymbol{\lambda}}(t) = -H_{\mathbf{x}}(\mathbf{x}^*(t),\mathbf{u}^*(t),\lambda_0^*(t),\boldsymbol{\lambda}^*(t)),\ a.e.\ [t_0,t_f^*]$$

$$with\ H(\mathbf{x}(t),\mathbf{u}(t),\lambda_0(t),\boldsymbol{\lambda}(t)) \equiv \lambda_0 L(\mathbf{x},\mathbf{u},t) + \boldsymbol{\lambda}^T\mathbf{F}(\mathbf{x},\mathbf{u},t)$$

$$\mathbf{u}^* = \arg\min_{\mathbf{u}\in U} H(\mathbf{x}^*(t),\mathbf{u}(t),\lambda_0^*(t),\boldsymbol{\lambda}^*(t))$$

$$\lambda_0^*(t) = const > 0, H(\mathbf{x}^*(t_t),\mathbf{u}^*(t_f),\lambda_0^*(t_f),\boldsymbol{\lambda}^*(t_f)) + \left(\varphi_t + \mathbf{v}^T\boldsymbol{\psi}_t\right)\Big|_{t=t_f} = 0$$

$$\boldsymbol{\lambda}^*(t_f) = \left(\varphi_x + \mathbf{v}^T\boldsymbol{\psi}_x\right)\Big|_{t=t_f}$$

$$\mathbf{v}^T\boldsymbol{\psi} = 0, \quad \mathbf{v} \geq 0$$

First Order Necessary Optimality Conditions

- Euler-Lagrange Equations:

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} = F_i(\mathbf{x},\mathbf{u},t) \qquad \dot{\lambda}_i = -\frac{\partial H}{\partial x_i} = -\frac{\partial L}{\partial x_i} - \lambda^T \frac{\partial \mathbf{F}}{\partial x_i} \ ; \ i = 1,\ldots,n$$

- with optimality condition

$$\frac{\partial H}{\partial u_j} = \frac{\partial L}{\partial u_j} + \lambda^T \frac{\partial \mathbf{F}}{\partial u_j} = 0 \quad j = 1,\ldots,l$$

$$t_0 \le t \le t_f$$

- and transfersality conditions:

$$\lambda(t_f) = \Phi_x^T = \left(\frac{\partial \varphi}{\partial x} + \upsilon^T \frac{\partial \psi}{\partial x} \right)^T \Bigg|_{t=t_f}$$

$$0 = (\Phi_t + H) \Big|_{t=t_f}$$

$$\Phi = \phi(\mathbf{x}_0, \mathbf{x}_f, t_f) + \upsilon^T \psi(\mathbf{x}_0, \mathbf{x}_f, t_f)$$

- First order conditions only guarantee that the solution is locally stationary.

- Legendre-Clebsch Condition: $H_{uu} \ge 0, \quad t_0 \le t \le t_f$

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Example 2: quadratic control

- Minimise

$$J = \int_{t_0}^{t_f} \frac{1}{2} u^2 dt$$

Subject to:

$$\dot{x} = v$$
$$\dot{v} = -x + u$$

With boundary conditions:

$$x(t_0) = 1$$
$$v(t_0) = 0$$
$$x(t_f) = 0$$
$$v(t_f) = 0$$

Example 2: quadratic control

- Hamiltonian

$$H = \frac{1}{2}u^2 + \lambda_x v + \lambda_v(-x+u)$$

with adjoint equations:

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = \lambda_v$$

$$\dot{\lambda}_v = -\frac{\partial H}{\partial v} = -\lambda_x$$

optimality condition:

$$\frac{\partial H}{\partial u} = u + \lambda_v$$

And transversality conditions:

$$\lambda_x(t_f) = \nu_x$$

$$\lambda_v(t_f) = \nu_v$$

From the adjoint equations and the optimal control condition we get:

$$\lambda_x(t) = A\sin t + B\cos t$$

$$\lambda_v(t) = B\sin t - A\cos t$$

$$u = A\cos t - B\sin t$$

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Example 2: quadratic control

and if one replaces the control law into the dynamic equation:

$$\dot{x} = v$$
$$\dot{v} = -x + A\cos t - B\sin t$$

which is a harmonic oscillator with a periodic forcing term:

$$\ddot{x} = -x + A\cos t - B\sin t$$

The solution is of the form:

$$x = C\cos t + D\sin t + \frac{A}{2}\cos t - \frac{B}{2}\sin t$$

And the four constants can be determined so that the boundary conditions are satisfied.

Optimal Control Problem with Mixed Path Constraints

- Consider the optimal control problem:

$$\min_{\mathbf{u}\in U} J(\mathbf{u},t_f) = \varphi\left(\mathbf{x}(t_0),\mathbf{x}(t_f),t_0,t_f\right) + \int_{t_0}^{t_f} L(\mathbf{x}(t),\mathbf{u}(t),t)dt$$

$$\begin{aligned}&\textit{subject to}:\\ &\dot{\mathbf{x}}(t) = \mathbf{F}(t,\mathbf{x}(t),\mathbf{u}(t))\\ &\boldsymbol{\psi}(\mathbf{x}(t_0),\mathbf{x}(t_f)) \geq 0\\ &\mathbf{G}(t,\mathbf{x}(t),\mathbf{u}(t)) \geq 0\end{aligned}$$

- Form the Lagrangian:

$$\begin{aligned}&\Lambda(\mathbf{u},t_f) = H + \boldsymbol{\mu}^T\mathbf{G}\\ &\textit{with}:\\ &H(\mathbf{x}(t),\mathbf{u}(t),\lambda_0(t),\boldsymbol{\lambda}(t)) \equiv \lambda_0 L(\mathbf{x},\mathbf{u},t) + \boldsymbol{\lambda}^T\mathbf{F}(\mathbf{x},\mathbf{u},t)\end{aligned}$$

- with Lagrangian multipliers $\boldsymbol{\mu}$

Maximum Principle with Maxed Path Constraints

If $(\mathbf{u}^*, t_f^*) \in \hat{C}[t_0,T]^q \times [t_0,T)$ is optimal, with response $\mathbf{x}^* \in \hat{C}^1[t_0,T]^n$, being $\hat{C}$ and $\hat{C}^1$ the spaces of piecewise continuous and piecewise continuous and differentiable functions respectively. Then, there exist $\left(\lambda_0^*, \boldsymbol{\lambda}^*\right) \in \hat{C}^1\left[t_0,T\right]^{n+1}$, $\boldsymbol{\mu} \in \hat{C}\left[t_0,T\right]^m$ and $\mathbf{v}^* \in \Re^{n_b}$ such that:

$$\left[\lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\mu}^*(t)\right] \neq [0,0], t_0 \leq t \leq t_f^*$$

$$\dot{\lambda}_0(t) = 0, \quad \dot{\boldsymbol{\lambda}}(t) = -\Lambda_{\mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\mu}^*(t)),$$

$$\dot{\mathbf{x}}(t) = \Lambda_{(\lambda_0,\boldsymbol{\lambda})}(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\mu}^*(t)), \; a.e. \; [t_0, t_f^*]$$

$$\mathbf{u}^* = \arg\min_{\mathbf{u} \in U^*} H(\mathbf{x}^*(t), \mathbf{u}(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t))$$

$$U^*(\mathbf{x}^*, t) = \left\{\mathbf{u} \in \Re^q \mid \mathbf{G}(\mathbf{x}^*, \mathbf{u}, t) \geq 0\right\}$$

$$\boldsymbol{\mu}^*(t)^T \mathbf{G} = 0; \boldsymbol{\mu}^* \geq 0;$$

$$\lambda_0^*(t) = const > 0, H(\mathbf{x}^*(t_t), \mathbf{u}^*(t_f), \lambda_0^*(t_f), \boldsymbol{\lambda}^*(t_f)) + \left(\varphi_t + \mathbf{v}^T \boldsymbol{\psi}_t\right)\Big|_{t=t_f} = 0$$

$$\boldsymbol{\lambda}^*(t_f) = \left(\varphi_x + \mathbf{v}^T \boldsymbol{\psi}_x\right)\Big|_{t=t_f}$$

$$\mathbf{v}^T \boldsymbol{\psi} = 0, \quad \mathbf{v} \geq 0$$

First Order Necessary Optimality Conditions with Mixed Path Constraints

- Euler-Lagrange equations:

$$\dot{x}_i = \frac{\partial \Lambda}{\partial \lambda_i} = F_i(\mathbf{x}, \mathbf{u}, t)$$

$$\dot{\lambda}_i = -\frac{\partial \Lambda}{\partial x_i} = -\frac{\partial L}{\partial x_i} - \lambda^T \frac{\partial \mathbf{F}}{\partial x_i} - \mu^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial x_i}; \quad i = 1, \ldots, n$$

- with Optimality condition:

$$\boxed{\frac{\partial \Lambda}{\partial u_j} = \frac{\partial L}{\partial u_j} + \lambda^T \frac{\partial \mathbf{F}}{\partial u_j} + \mu^T \frac{\partial \mathbf{G}}{\partial u_j} = 0 \quad j = 1, \ldots, l}$$

$$\left.\begin{aligned} &\lambda(t_f) = \Phi_x^T = \left(\frac{\partial \varphi}{\partial x} + \upsilon^T \frac{\partial \psi}{\partial x} \right)^T \Bigg|_{t=t_f} \\ &0 = (\Phi_t + H) \Big|_{t=t_f} \\ &\Phi = \phi(\mathbf{x}_0, \mathbf{x}_f, t_f) + \upsilon^T \psi(\mathbf{x}_0, \mathbf{x}_f, t_f) \end{aligned}\right\} \text{Transversality conditions}$$

- and second order condition:

$$\boxed{\Lambda_{uu} > 0}$$

Example of Solution

- The extended Hamiltonian is

$$\Lambda = \lambda_x v + \lambda_v \left(\frac{T - D}{m} - g \right) - \lambda_m \frac{T}{g_0 I_{sp}} + \boxed{\mu_1 T + \mu_2 (T_{\max} - T)}$$

where only one of the two conditions on the thrust is true at any one time.

The dynamic equations are:

$$\dot{x} = \frac{\partial \Lambda}{\partial \lambda_x} = v$$

$$\dot{v} = \frac{\partial \Lambda}{\partial \lambda_v} = \frac{T - D}{m} - g;$$

$$\dot{m} = \frac{\partial \Lambda}{\partial \lambda_m} = -\frac{T}{g_0 I_{sp}}$$

and the adjoint equations are:

$$\dot{\lambda}_x(t) = -\frac{\partial \Lambda}{\partial x} = -\lambda_v \frac{1}{m} \frac{\partial D}{\partial x}; \quad \dot{\lambda}_v(t) = -\frac{\partial \Lambda}{\partial v} = -\lambda_x - \lambda_v \frac{1}{m} \frac{\partial D}{\partial v}; \quad \dot{\lambda}_m(t) = -\frac{\partial \Lambda}{\partial m} = \lambda_v \frac{T - D}{m^2}$$

Example of Solution

- The optimal control condition writes

$$\frac{\partial \Lambda}{\partial T} = \frac{\lambda_v}{m} - \frac{\lambda_m}{g_0 I_{sp}} + \mu_1 - \mu_2 = 0$$

where again only one of the two μ is true at any one time, and the transversality conditions are:

$$\lambda_x(t_f) = \upsilon_x \frac{\partial \psi}{\partial x} = \upsilon_x$$

$$\lambda_v(t_f) = \frac{\partial \phi}{\partial v} = 1$$

$$\lambda_m(t_f) = 0$$

If one looks at the optimality condition can define the switching function as:

$$I = \frac{\lambda_v}{m} - \frac{\lambda_m}{g_0 I_{sp}} > 0 \Rightarrow \mu_2 > 0, \mu_1 = 0 \Rightarrow T = T_{\max}$$

$$I = \frac{\lambda_v}{m} - \frac{\lambda_m}{g_0 I_{sp}} < 0 \Rightarrow \mu_1 > 0, \mu_2 = 0 \Rightarrow T = 0$$

$$I = \frac{\lambda_v}{m} - \frac{\lambda_m}{g_0 I_{sp}} = 0 \Rightarrow \text{ singular arc}$$

Treatment of Singular Arcs

- Along singular arcs we have that:

$$H_u = 0$$

Is satisfied for any admissible control. A solution is to differentiate a minimum of *p* times with respect to time until one obtains:

$$\frac{d^p}{dt^p} H_u = 0$$

$$\frac{\partial}{\partial u}\left(\frac{d^p}{dt^p} H_u\right) \neq 0$$

The second equation provides the required control law while *p-1* is the order of the singular arc.

Optimal Control Problem with State Path Constraints

- Consider the optimal control problem:

$$\min_{\mathbf{u}\in U} J(\mathbf{u},t_f)=\varphi\left(\mathbf{x}(t_0),\mathbf{x}(t_f),t_0,t_f\right)+\int_{t_0}^{t_f} L(\mathbf{x}(t),\mathbf{u}(t),t)dt$$

$$subject\ to:$$

$$\dot{\mathbf{x}}(t)=\mathbf{F}(t,\mathbf{x}(t),\mathbf{u}(t))$$

$$\boldsymbol{\psi}(\mathbf{x}(t_0),\mathbf{x}(t_f))\geq 0$$

$$\mathbf{G}(t,\mathbf{x}(t))\geq 0$$

- In this case the path constraint can not be directly used to form a Lagrangian function. In fact suppose that **G**≥0 is a path constraint on the position. This constraint does not provide any information on the velocity and acceleration at the contact point **G**=0. As a consequence there is no information on the correct control to be applied. This is particularly important when pure state path constraints are introduced via a direct approach.
- The correct way to introduce a pure state path constraint is to differentiate *p* times with respect to time till the control appears explicitly:

$$\frac{\partial}{\partial \mathbf{u}}\frac{d^p\mathbf{G}}{dt^p}\neq 0$$

Optimal Control Problem with State Path Constraints

- Once the control appears explicitly in the constraints, one can form the Lagrangian (indirect adjoining approach):

$$\Lambda(\mathbf{u},t_f) = H + \boldsymbol{\eta}^T \frac{d^p\mathbf{G}}{dt^p}$$

$$with:$$

$$H(\mathbf{x}(t),\mathbf{u}(t),\lambda_0(t),\boldsymbol{\lambda}(t)) \equiv \lambda_0 L(\mathbf{x},\mathbf{u},t) + \boldsymbol{\lambda}^T\mathbf{F}(\mathbf{x},\mathbf{u},t)$$

- For example for p=1 and **G**≥0 one has:

$$\frac{d\mathbf{G}(\mathbf{x}^*,t)}{dt} \geq 0$$

$$\mathbf{G}(\mathbf{x}^*,t) = 0$$

- :At a contact point τ the second constraints causes a discontinuity in adjoint variables and Hamiltonian:

$$\lambda^*(\tau^-) = \lambda^*(\tau^+) + \theta(\tau)h_x(\tau,\mathbf{x}^*)$$

$$H(\tau^-) = H(\tau^+) - \theta(\tau)h_t(\tau,\mathbf{x}^*)$$

- with θ another Lagrange multiplier

Maximum Principle with State Path Constraints

If $(\mathbf{u}^*, t_f^*) \in \hat{C}[t_0,T]^q \times [t_0,T)$ is optimal, with response $\mathbf{x}^* \in \hat{C}^1[t_0,T]^n$, being $\hat{C}$ and $\hat{C}^1$ the spaces of piecewise continuous and piecewise continuous and differentiable functions respectively. Then, there exist $\left(\lambda_0^*, \boldsymbol{\lambda}^*\right) \in \hat{C}^1\left[t_0,T\right]^{n+1}$, $\boldsymbol{\eta} \in \hat{C}\left[t_0,T\right]^m$, $\mathbf{v}^* \in \Re^{n_b}$, and $\boldsymbol{\theta}(\tau) \in \Re^{n_c}$ such that:

$$\left[\lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\eta}^*(t)\right] \neq [0,0], t_0 \le t \le t_f^*$$

$$\dot{\lambda}_0(t) = 0, \quad \dot{\boldsymbol{\lambda}}(t) = -\Lambda_{\mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\eta}^*(t)),$$

$$\dot{\mathbf{x}}(t) = \Lambda_{(\lambda_0,\boldsymbol{\lambda})}(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\eta}^*(t)), \ a.e.\ [t_0, t_f^*]$$

$$\mathbf{u}^* = \arg\min_{\mathbf{u}\in U_1^*} H(\mathbf{x}^*(t), \mathbf{u}(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t))$$

$$U_1^*(\mathbf{x}^*, t) = \left\{\mathbf{u} \in \Re^q \mid \frac{d\mathbf{G}}{dt}(\mathbf{x}^*, \mathbf{u}, t) \ge 0, \mathbf{G}(\mathbf{x}^*, \mathbf{u}, t) = 0\right\}$$

$$\boldsymbol{\eta}^*(t)^T \mathbf{G} = 0; \boldsymbol{\eta}^* \ge 0; \dot{\boldsymbol{\eta}}^* \le 0$$

$$\lambda_0^*(t) = const > 0, H(\mathbf{x}^*(t_t), \mathbf{u}^*(t_f), \lambda_0^*(t_f), \boldsymbol{\lambda}^*(t_f)) + \left(\varphi_t + \mathbf{v}^T \boldsymbol{\psi}_t\right)\Big|_{t=t_f} = 0$$

$$\boldsymbol{\lambda}^*(t_f) = \left(\varphi_x + \mathbf{v}^T \boldsymbol{\psi}_x\right)\Big|_{t=t_f}$$

$$\mathbf{v}^T \boldsymbol{\psi} = 0, \quad \mathbf{v} \ge 0$$

$$\boldsymbol{\lambda}^*(\tau^-)^T = \boldsymbol{\lambda}^*(\tau^+)^T + \boldsymbol{\theta}(\tau)^T \mathbf{G}_x(\tau, \mathbf{x}^*)$$

$$H(\tau^-) = H(\tau^+) - \boldsymbol{\theta}(\tau)^T \mathbf{G}_t(\tau, \mathbf{x}^*)$$

$$\boldsymbol{\theta}^*(\tau)\mathbf{G}(\tau, \mathbf{x}^*(\tau)) = 0; \theta^*(\tau) \ge 0; \theta(\tau) \le \eta(\tau^+)$$

Some References

- **Bryson A E, Jr. & Ho Y C.** *Applied optimal control: optimization, estimation, and control.* Waltham, MA: Blaisdell, 1969. 481 p.
- **Betts J.T.** *Practical Methods for Optimal Control and Estimation Using Nonlinear Programming*. Advances in Design and Control, SIAM 2001

FINITE ELEMENTS IN TIME

Previous Works on Finite Elements in Time for Optimal Control

Since early '90s Finite Elements in Time used for Indirect Transcription:

- Hodges, D. H. and Bless, R. R., "Weak Hamiltonian Finite Element Method for Optimal Control Problems," Journal of Guidance, Control, and Dynamics, Vol. 14, No. 1, **1991**, pp. 148–156.
- Hodges, D. H., Bless, R. R., Calise, A. J., and Leung, M., "Finite Element Method for Optimal Guidance of an Advanced Launch Vehicle," Journal of Guidance, Control, and Dynamics, Vol. 15, No. 3, **1992**, pp. 664–671.
- Bottasso, C. and Ragazzi, A., "Finite Element and Runge-Kutta Methods for Boundary-Value and Optimal Control Problems," Journal of Guidance, Control, and Dynamics, Vol. 23, No. 4, **2000**, pp. 749–751.

Late '90s Finite Elements in Time on Spectral Basis for Direct Transcription:

- Vasile M., Bottasso C.L., Finzi A.E., Lunar Orbital Dynamics by Finite Element in Time Method, Aerotecnica Missili e Spazio Vol. 75. Numero 3/4, Luglio-Dicembre **1996.**
- Finzi A., Vasile M. Numerical Solutions for Lunar Orbits. IAF-97-A.5.08, 48th International Astronautical Congress, October 6-10, **1997**/Turin, Italy
- Vasile, M. and Finzi, A., "Direct Lunar Descent Optimisation by Finite Elements in Time Approach," Journal of Mechanics and Control, Vol. 1, No. 1, **2000**.
- Vasile, M. and Bernelli-Zazzera, F., "Optimizing Low-Thrust and Gravity Assist Maneuvres to Design Interplanetary Trajectories," The Journal of the Astronautical Sciences, Vol. 51, No. 1, 2003, January-March **2003**.

Direct Finite Element Transcription (DFET)

- Strong solution of the differential equations:

$$\int_{t_0}^{t_f} \left(\dot{\mathbf{x}} - \mathbf{F}(\mathbf{x},\mathbf{u})\right) dt = \mathbf{x}^b \Big|_{t_0}^{t_f}$$

- **Weak solution of the differential equations:**

$$\int_{t_0}^{t_f} \mathbf{w}^T \left(\dot{\mathbf{x}} - \mathbf{F}(\mathbf{x},\mathbf{u})\right) dt = \mathbf{w}^T (\mathbf{x} - \mathbf{x}^b)\Big|_{t_0}^{t_f}$$

- The solution **x** and control **u** satisfy the differential equations in a weak sense, i.e. with respect to the test functions **w**.

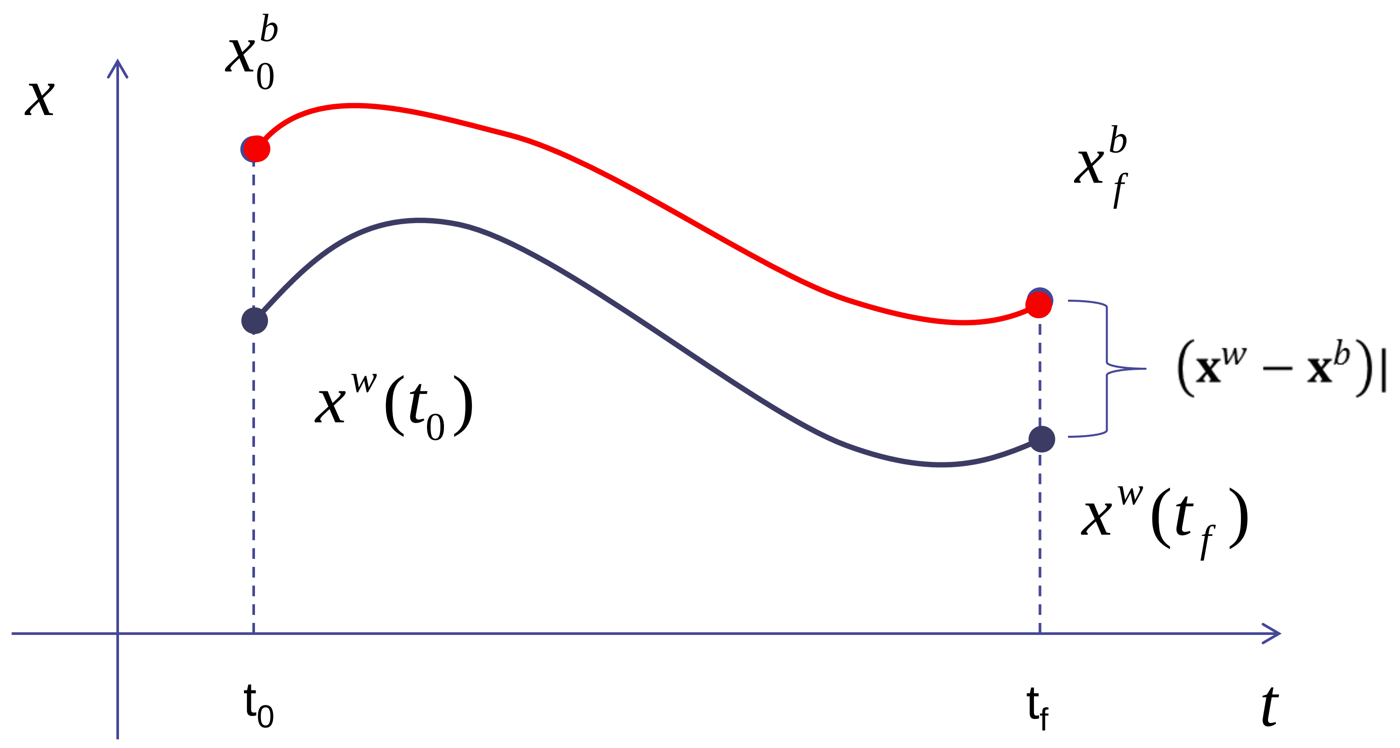
- Boundary conditions are not exactly satisfied but are satisfied with respect to the test functions **w**.

Direct Finite Element Transcription (DFET)

- Strong vs. Weak solution of the differential equations:

$$\int_{t_0}^{t_f} \left(\dot{\mathbf{x}} - \mathbf{F}(\mathbf{x},\mathbf{u})\right) dt = \mathbf{x}^b \Big|_{t_0}^{t_f}$$

$$\int_{t_0}^{t_f} \mathbf{w}^T \left(\dot{\mathbf{x}}^w - \mathbf{F}(\mathbf{x}^w,\mathbf{u}^w)\right) dt = \mathbf{w}^T (\mathbf{x}^w - \mathbf{x}^b)\Big|_{t_0}^{t_f}$$



Direct Finite Element Transcription (DFET)

- Partial integration of the differential terms:

$$\int_{t_0}^{t_f} \mathbf{w}^T\dot{\mathbf{x}} - \mathbf{w}^T\mathbf{F}(\mathbf{x},\mathbf{u})dt = \mathbf{w}^T\mathbf{x} - \mathbf{w}^T\mathbf{x}^b\Big|_{t_0}^{t_f}$$

$$\int_{t_0}^{t_f} \mathbf{w}^T\dot{\mathbf{x}} - \mathbf{w}^T\mathbf{F}(\mathbf{x},\mathbf{u})dt - \mathbf{w}^T\mathbf{x}\Big|_{t_0}^{t_f} = -\mathbf{w}^T\mathbf{x}^b\Big|_{t_0}^{t_f}$$

$$\int_{t_0}^{t_f} \mathbf{w}^T\dot{\mathbf{x}} - \frac{d}{dt}\left(\mathbf{w}^T\mathbf{x}\right) - \mathbf{w}^T\mathbf{F}(\mathbf{x},\mathbf{u})dt = -\mathbf{w}^T\mathbf{x}^b\Big|_{t_0}^{t_f}$$

$$\int_{t_0}^{t_f} \dot{\mathbf{w}}^T\mathbf{x} + \mathbf{w}^T\mathbf{F}(\mathbf{x},\mathbf{u})dt = \mathbf{w}^T\mathbf{x}^b\Big|_{t_0}^{t_f}$$

- The integral can be solved by polynomial representation of the states **x**, control **u** and test functions **w** and Gauss quadrature.
- The choice of the test function **w** can be such that the projection is orthogonal.

Direct Finite Element Transcription (DFET)

$$\int_{t_0}^{t_f} \dot{\mathbf{w}}^T \mathbf{x} + \mathbf{w}^T \mathbf{F}(\mathbf{x}, \mathbf{u}) dt = \mathbf{w}^T \mathbf{x}^b \Big|_{t_0}^{t_f}$$

- Polynomial representation of **x**, **u** and **w**:

$$\begin{Bmatrix} \mathbf{x} \\ \mathbf{u} \end{Bmatrix} = \sum_{s=1}^{p} f_s(t) \begin{Bmatrix} \mathbf{x}_s \\ \mathbf{u}_s \end{Bmatrix} \qquad \mathbf{w} = \sum_{s=1}^{p+1} g_s(t) \mathbf{w}_s$$

- Where $\mathbf{x}_s$, $\mathbf{u}_s$ and $\mathbf{w}_s$ are discrete nodes and f_s and g_s are polynomials of order p-1 and p respectively.
- Note that in principle f_s and g_s can be any arbitrary function generated on any arbitrary base.

$$\int_{t_0}^{t_f} \left(\sum_{s=1}^{p+1} \mathbf{w}_s \dot{g}_s \right)^T \sum_{s=1}^{p} \mathbf{x}_s f_s + \left(\sum_{s=1}^{p+1} \mathbf{w}_s g_s \right)^T \mathbf{F}(\mathbf{x}_s, \mathbf{u}_s, t) dt = w_{p+1} g_{p+1}(t_f) \mathbf{x}_{t_f} - w_1 g_1(t_0) \mathbf{x}_{t_0}$$

Direct Finite Element Transcription (DFET)

$$\int_{t_0}^{t_f}\left(\sum_{s=1}^{p+1}\mathbf{w}_s\dot{g}_s\right)^T\sum_{s=1}^{p}\mathbf{x}_s f_s+\left(\sum_{s=1}^{p+1}\mathbf{w}_s g_s\right)^T\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,t)dt=w_{p+1}g_{p+1}(t_f)\mathbf{x}_{t_f}-w_1 g_1(t_0)\mathbf{x}_{t_0}$$

- The equation must hold true for any arbitrary $\mathbf{w}_s$, hence:

$$\int_{t_0}^{t_f}\dot{g}_1(t)\sum_{s=1}^{p}\mathbf{x}_s f_s(t)+g_1(t)\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,t)dt=-\mathbf{x}_{t_0}$$

$$\dots$$

$$\int_{t_0}^{t_f}\dot{g}_j(t)\sum_{s=1}^{p}\mathbf{x}_s f_s(t)+g_j\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,t)dt=0$$

$$\dots$$

$$\int_{t_0}^{t_f}\dot{g}_{p+1}(t)\sum_{s=1}^{p}\mathbf{x}_s f_s(t)+g_{p+1}(t)\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,t)dt=\mathbf{x}_{t_f}$$

Direct Finite Element Transcription (DFET)

- Now assume we wanted to use Gauss quadrature formulas to solve the integrals, then we would have:

$$\sum_{k=1}^{p}\omega_k\left[\dot{g}_1(\tau_k)\sum_{s=1}^{p}\mathbf{x}_s f_s(\tau_k)+g_1(\tau_k)\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,\tau_k)\right]=-\mathbf{x}_{t_0}$$

$$\ldots$$

$$\sum_{k=1}^{p}\omega_k\left[\dot{g}_j(\tau_k)\sum_{s=1}^{p}\mathbf{x}_s f_s(\tau_k)+g_j(\tau_k)\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,\tau_k)\right]=0$$

$$\ldots$$

$$\sum_{k=1}^{p}\omega_k\left[\dot{g}_{p+1}(\tau_k)\sum_{s=1}^{p}\mathbf{x}_s f_s(\tau_k)+g_{p+1}(\tau_k)\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,\tau_k)\right]=\mathbf{x}_{t_f}$$

- Where τ_k are Gauss points and τ_k are Gauss weights.

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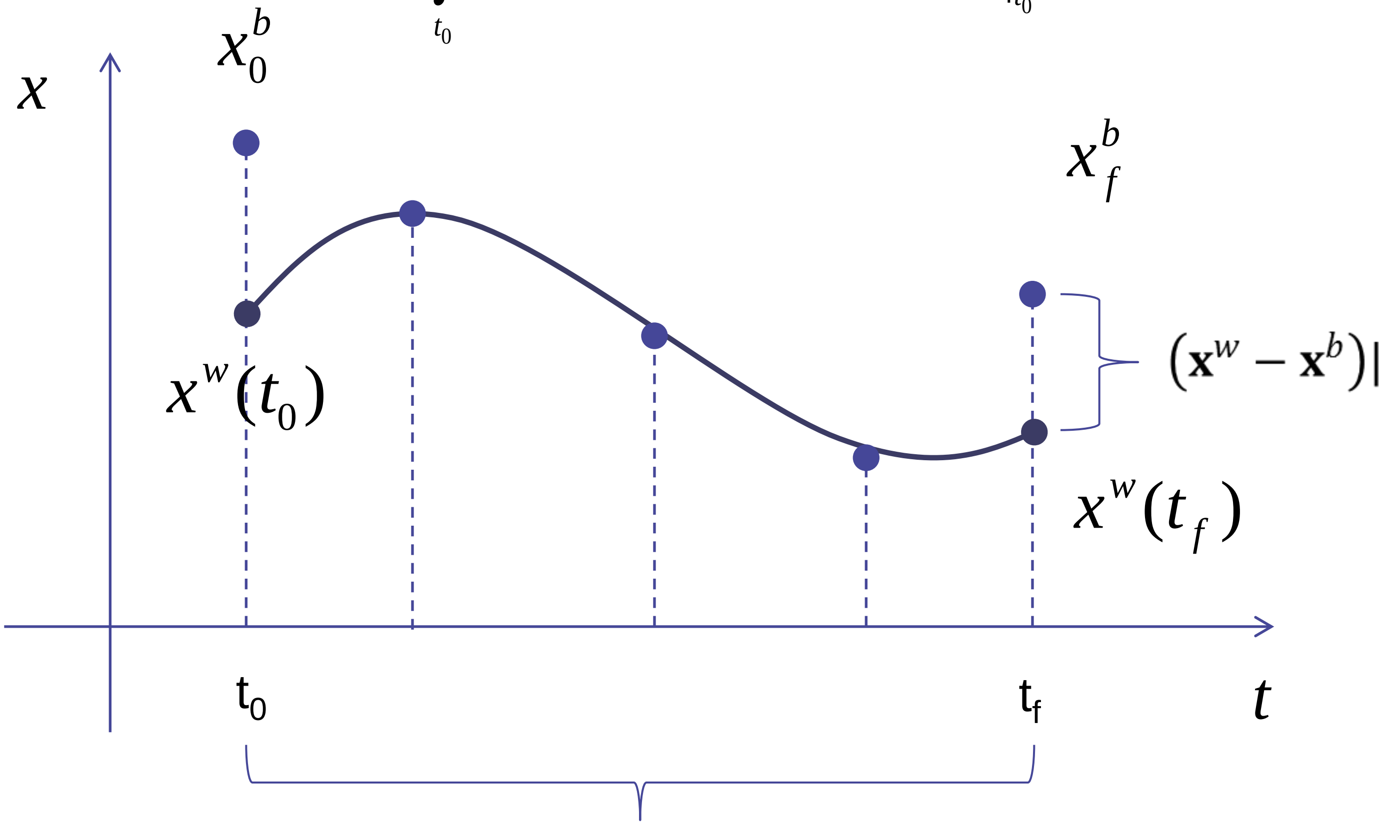
Direct Finite Element Transcription (DFET)

- Discretised Weak solution of the differential equations:

$$\int_{t_0}^{t_f} \dot{\mathbf{w}}^T \mathbf{x} + \mathbf{w}^T \mathbf{F}(\mathbf{x}, \mathbf{u}) dt = \mathbf{w}^T \mathbf{x}^b \Big|_{t_0}^{t_f}$$



Spectral basis: Gauss points used for integration and discretisation

Example of DFET Integration

- Let us consider the simple linear system with a constant *u*:

$$\dot{x} = v$$
$$\dot{v} = -x + u$$

- In weak form we can write:

$$\int_{t_0}^{t_f} \dot{\mathbf{w}}^T \begin{bmatrix} x \\ v \end{bmatrix} + \mathbf{w}^T \begin{bmatrix} v \\ -x+u \end{bmatrix} dt = \mathbf{w}^T \begin{bmatrix} x \\ v \end{bmatrix}^b \Bigg|_{t_0}^{t_f}$$

- And assume we use polynomials of order 0 for states and controls and of order 1 for the test functions:

$$f = 1; w = w_1 \frac{(\tau_f - \tau)}{(\tau_f - \tau_0)} + w_2 \frac{(\tau - \tau_0)}{(\tau_f - \tau_0)}; \dot{w} = -w_1 \frac{1}{(\tau_f - \tau_0)} + w_2 \frac{1}{(\tau_f - \tau_0)}$$

Example of DFET Integration

- The weak solution becomes:

$$\int_{t_0}^{t_f}\left(-w_1\frac{1}{(\tau_f-\tau_0)}+w_2\frac{1}{(\tau_f-\tau_0)}\right)\begin{bmatrix}x_s\\v_s\end{bmatrix}+\left(w_1\frac{(\tau_f-\tau)}{(\tau_f-\tau_0)}+w_2\frac{(\tau-\tau_0)}{(\tau_f-\tau_0)}\right)\begin{bmatrix}v_s\\-x_s+u_s\end{bmatrix}dt=w_2\begin{bmatrix}x\\v\end{bmatrix}_f^b-w_1\begin{bmatrix}x\\v\end{bmatrix}_0^b$$

- Because they need to be true for any arbitrary weight *w* we can derive the two equations:

$$\int_{t_0}^{t_f}-\frac{1}{(\tau_f-\tau_0)}\begin{bmatrix}x_s\\v_s\end{bmatrix}+\frac{(\tau_f-\tau)}{(\tau_f-\tau_0)}\begin{bmatrix}v_s\\-x_s+u_s\end{bmatrix}dt=-\begin{bmatrix}x\\v\end{bmatrix}_0^b$$

$$\int_{t_0}^{t_f}\frac{1}{(\tau_f-\tau_0)}\begin{bmatrix}x_s\\v_s\end{bmatrix}+\frac{(\tau-\tau_0)}{(\tau_f-\tau_0)}\begin{bmatrix}v_s\\-x_s+u_s\end{bmatrix}dt=\begin{bmatrix}x\\v\end{bmatrix}_f^b$$

- We can now take Gauss integration formulas to solve the integral:

$$\sigma_1\left(-\frac{1}{\Delta t}\begin{bmatrix}x_s\\v_s\end{bmatrix}+\frac{\tau_f-\tau}{\Delta t}\begin{bmatrix}v_s\\-x_s+u_s\end{bmatrix}\right)\frac{\Delta t}{2}=-\begin{bmatrix}x\\v\end{bmatrix}_0^b$$

$$\sigma_1\left(\frac{1}{\Delta t}\begin{bmatrix}x_s\\v_s\end{bmatrix}+\frac{\tau-\tau_0}{\Delta t}\begin{bmatrix}v_s\\-x_s+u_s\end{bmatrix}\right)\frac{\Delta t}{2}=\begin{bmatrix}x\\v\end{bmatrix}_f^b$$

Example of DFET Integration

- The Gauss weight for a single point is equal to 2, therefore we have:

$$2\left(-\frac{1}{\Delta t}\begin{bmatrix} x_s \\ v_s \end{bmatrix} + \frac{\Delta t/2}{\Delta t}\begin{bmatrix} v_s \\ -x_s + u_s \end{bmatrix}\right)\frac{\Delta t}{2} = -\begin{bmatrix} x \\ v \end{bmatrix}_0^b \quad \Rightarrow \quad -\begin{bmatrix} x_s \\ v_s \end{bmatrix} + \frac{\Delta t}{2}\begin{bmatrix} v_s \\ -x_s + u_s \end{bmatrix} = -\begin{bmatrix} x \\ v \end{bmatrix}_0^b$$

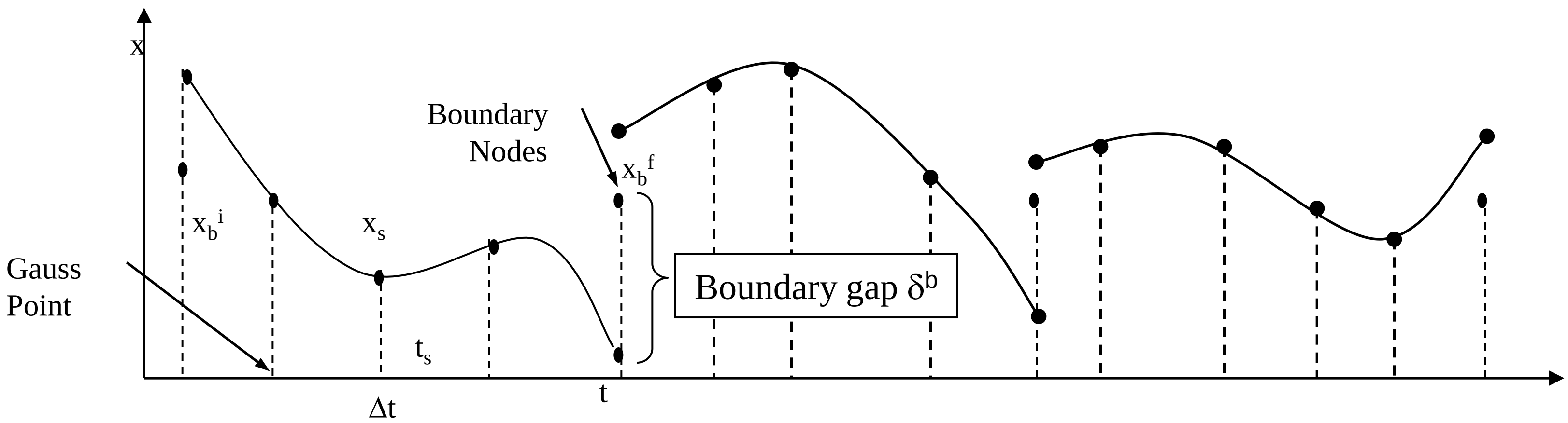
$$2\left(\frac{1}{\Delta t}\begin{bmatrix} x_s \\ v_s \end{bmatrix} + \frac{\Delta t/2}{\Delta t}\begin{bmatrix} v_s \\ -x_s + u_s \end{bmatrix}\right)\frac{\Delta t}{2} = \begin{bmatrix} x \\ v \end{bmatrix}_f^b \quad \Rightarrow \quad \begin{bmatrix} x_s \\ v_s \end{bmatrix} + \frac{\Delta t}{2}\begin{bmatrix} v_s \\ -x_s + u_s \end{bmatrix} = \begin{bmatrix} x \\ v \end{bmatrix}_f^b$$

- And if we sum the second to the first we get:

$$\begin{bmatrix} x \\ v \end{bmatrix}_f^b = \begin{bmatrix} x \\ v \end{bmatrix}_0^b + \Delta t\begin{bmatrix} v_s \\ -x_s + u_s \end{bmatrix}$$

- Note that for nonlinear function this is an implicit scheme.

Direct Finite Element Transcription (DFET)



- The **Time domain** is decomposed in **finite elements** leading to a **polynomial development** of the solution on **spectral basis** (Gauss Points)

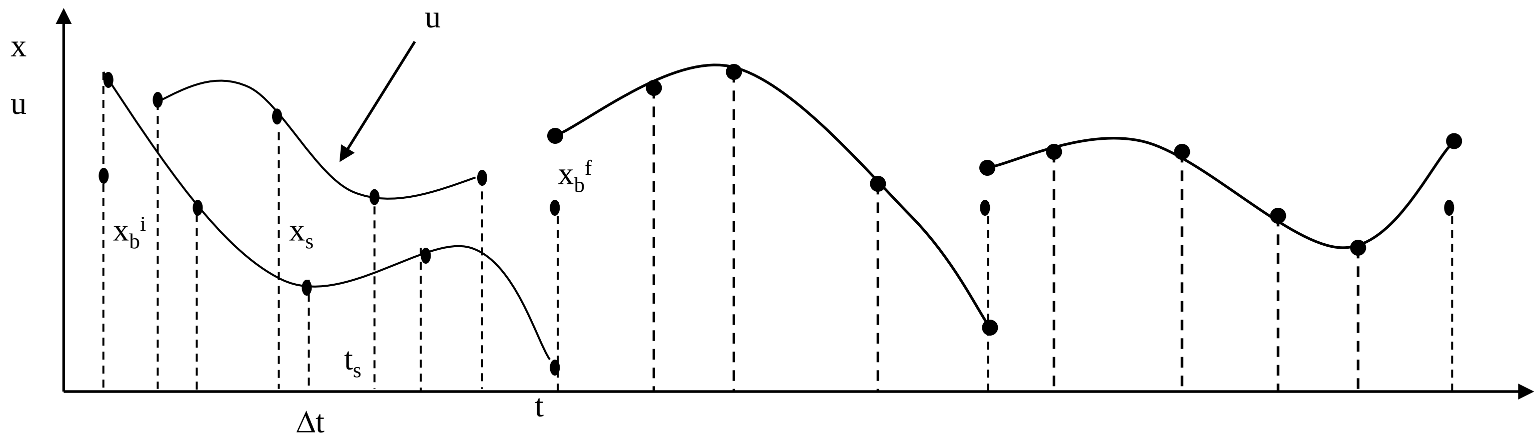
$$D = \bigcup_{i=1}^{N} D_i(t_{i-1}, t_i)$$

- **Differential constraints** are expressed in **weak form** leading to discontinuities at boundaries

$$\int_{t_0}^{t_f} \mathbf{w}^T \left(\dot{\mathbf{x}} - \mathbf{F}(\mathbf{x},\mathbf{u})\right) dt = \mathbf{w}^T \left(\mathbf{x} - \mathbf{x}^b\right)\Big|_{t_0}^{t_f}$$

- Gauss formulas for the solution of the integral: high Integration order 2n = 2k+2

Direct Finite Element Transcription (DFET)



- The controls are discretised using the Guass points used for integration.
- The controls can be discontinuous at the boundaries.

- The integration error within an element is absorbed into the discontinuity at the boundaries.

DFET in Summary

- Representation of controls, states and weights on spectral basis.

$$f_s \in P^{p-1}(D_i)\,;\;\; g_s \in P^{p}(D_i) \longrightarrow \begin{Bmatrix}\mathbf{x}\\ \mathbf{u}\end{Bmatrix} = \sum_{s=1}^{p} f_s(t)\begin{Bmatrix}\mathbf{x}_s\\ \mathbf{u}_s\end{Bmatrix}$$

$$\mathbf{w} = \sum_{s=1}^{p+1} g_s(t)\mathbf{w}_s$$

- Same bases for the discretisation of objective function and algebraic constraints:

$$J = \phi(\mathbf{x}_0^b, \mathbf{x}_f^b, t_f) + \sum_{i=1}^{N}\sum_{k=1}^{q} \sigma_k L[\mathbf{x}(\tau_k), \mathbf{u}(\tau_k), \mathbf{p}, \tau_k]\frac{\Delta t_i}{2}$$

$$\mathbf{G}(\mathbf{x}_s(\xi_k), \mathbf{u}_s(\xi_k), \xi_k) \geq 0$$

- Differential equations are transformed into algebraic equations:

$$\sum_{k=1}^{q} \sigma_k \left[\dot{\mathbf{w}}(\tau_k)^T \mathbf{x}(\tau_k) + \mathbf{w}(\tau_k)^T \mathbf{F}(\tau_k)\frac{\Delta t_i}{2}\right] - \mathbf{w}_{p+1}^T \mathbf{x}_i^b + \mathbf{w}_1^T \mathbf{x}_{i-1}^b = 0$$

DFET Assembling Process

- :

$$\int_{t_i}^{t_{i+1}} \dot{g}_1(t)\sum_{s=1}^{p}\mathbf{x}_s f_s(t) + g_1(t)\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,t)dt = -\mathbf{x}_i$$

...

$$\int_{t_i}^{t_{i+1}} \dot{g}_j(t)\sum_{s=1}^{p}\mathbf{x}_s f_s(t) + g_j\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,t)dt = 0$$

...

$$\int_{t_i}^{t_{i+1}} \dot{g}_{p+1}(t)\sum_{s=1}^{p}\mathbf{x}_s f_s(t) + g_{p+1}(t)\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,t)dt = \mathbf{x}_{i+1}$$

The end boundary node of one element must be equal to the beginning boundary node of the following element

$$\int_{t_{i+1}}^{t_{i+2}} \dot{g}_1(t)\sum_{s=1}^{p}\mathbf{x}_s f_s(t) + g_1(t)\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,t)dt = -\mathbf{x}_{i+1}$$

...

$$\int_{t_{i+1}}^{t_{i+2}} \dot{g}_j(t)\sum_{s=1}^{p}\mathbf{x}_s f_s(t) + g_j\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,t)dt = 0$$

...

$$\int_{t_{i+1}}^{t_{i+2}} \dot{g}_{p+1}(t)\sum_{s=1}^{p}\mathbf{x}_s f_s(t) + g_{p+1}(t)\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,t)dt = \mathbf{x}_{i+2}$$

The last equation of one element is matched to the first equation of the following element.
Boundary nodes disappear except for the two extremal nodes

Example of DFET Assembly

- Assume we have two elements:

$$-\begin{bmatrix} x_{s_1} \\ v_{s_1} \end{bmatrix} + \frac{\Delta t}{2}\begin{bmatrix} v_{s_1} \\ -x_{s_1} + u_{s_1} \end{bmatrix} = -\begin{bmatrix} x \\ v \end{bmatrix}_0^b \qquad -\begin{bmatrix} x_{s_2} \\ v_{s_2} \end{bmatrix} + \frac{\Delta t}{2}\begin{bmatrix} v_{s_2} \\ -x_{s_2} + u_{s_2} \end{bmatrix} = -\begin{bmatrix} x \\ v \end{bmatrix}_2^b$$

$$\begin{bmatrix} x_{s_1} \\ v_{s_1} \end{bmatrix} + \frac{\Delta t}{2}\begin{bmatrix} v_{s_1} \\ -x_{s_1} + u_{s_1} \end{bmatrix} = \begin{bmatrix} x \\ v \end{bmatrix}_1^b \qquad \begin{bmatrix} x_{s_2} \\ v_{s_2} \end{bmatrix} + \frac{\Delta t}{2}\begin{bmatrix} v_{s_2} \\ -x_{s_2} + u_{s_2} \end{bmatrix} = \begin{bmatrix} x \\ v \end{bmatrix}_f^b$$

- The end of the first element must be equal to the beginning of element 2:

$$-\begin{bmatrix} x_{s_1} \\ v_{s_1} \end{bmatrix} + \frac{\Delta t}{2}\begin{bmatrix} v_{s_1} \\ -x_{s_1} + u_{s_1} \end{bmatrix} = -\begin{bmatrix} x \\ v \end{bmatrix}_0^b$$

$$\begin{bmatrix} x_{s_1} \\ v_{s_1} \end{bmatrix} + \frac{\Delta t}{2}\begin{bmatrix} v_{s_1} \\ -x_{s_1} + u_{s_1} \end{bmatrix} = \begin{bmatrix} x_{s_2} \\ v_{s_2} \end{bmatrix} - \frac{\Delta t}{2}\begin{bmatrix} v_{s_2} \\ -x_{s_2} + u_{s_2} \end{bmatrix}$$

$$\begin{bmatrix} x_{s_2} \\ v_{s_2} \end{bmatrix} + \frac{\Delta t}{2}\begin{bmatrix} v_{s_2} \\ -x_{s_2} + u_{s_2} \end{bmatrix} = \begin{bmatrix} x \\ v \end{bmatrix}_f^b$$

Direct Finite Element Transcription (DFET)

- The optimal control problem is transformed into a nonlinear programming problem

$$J = \phi(\mathbf{x}_0^b, \mathbf{x}_f^b, t_f, \mathbf{p}) + \int_{t_i}^{t_f} L\,(\mathbf{x}, \mathbf{u}, \mathbf{p})dt$$

$$\dot{\mathbf{x}} - \mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) = 0$$

$$\mathbf{G}(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) \geq 0$$

$$\psi(\mathbf{x}_0^b, \mathbf{x}_f^b, \mathbf{p}, t)\Big|_{t_0}^{t_f} \geq 0$$

$$\min \quad J(\mathbf{y})$$

$$\mathbf{c}(\mathbf{y}) \geq 0$$

$$\mathbf{b}_l \leq \mathbf{y} \leq \mathbf{b}_u$$

$$\mathbf{y}=[\mathbf{x}_s,\mathbf{u}_s,\mathbf{x}^b{}_0,\mathbf{x}^b{}_f,t_0,t_f,\mathbf{p}]$$

Direct Finite Element Transcription (DFET)

Optimal Control Problem → **DFET** → NLP Problem → **SSQP** Sparse Sequential Quadratic Programming

NLP Problem

$$\min J(\mathbf{y})$$

$$\mathbf{c}(\mathbf{y}) \ge \mathbf{0}$$

$$\mathbf{b}_l \le \mathbf{y} \le \mathbf{b}_u$$

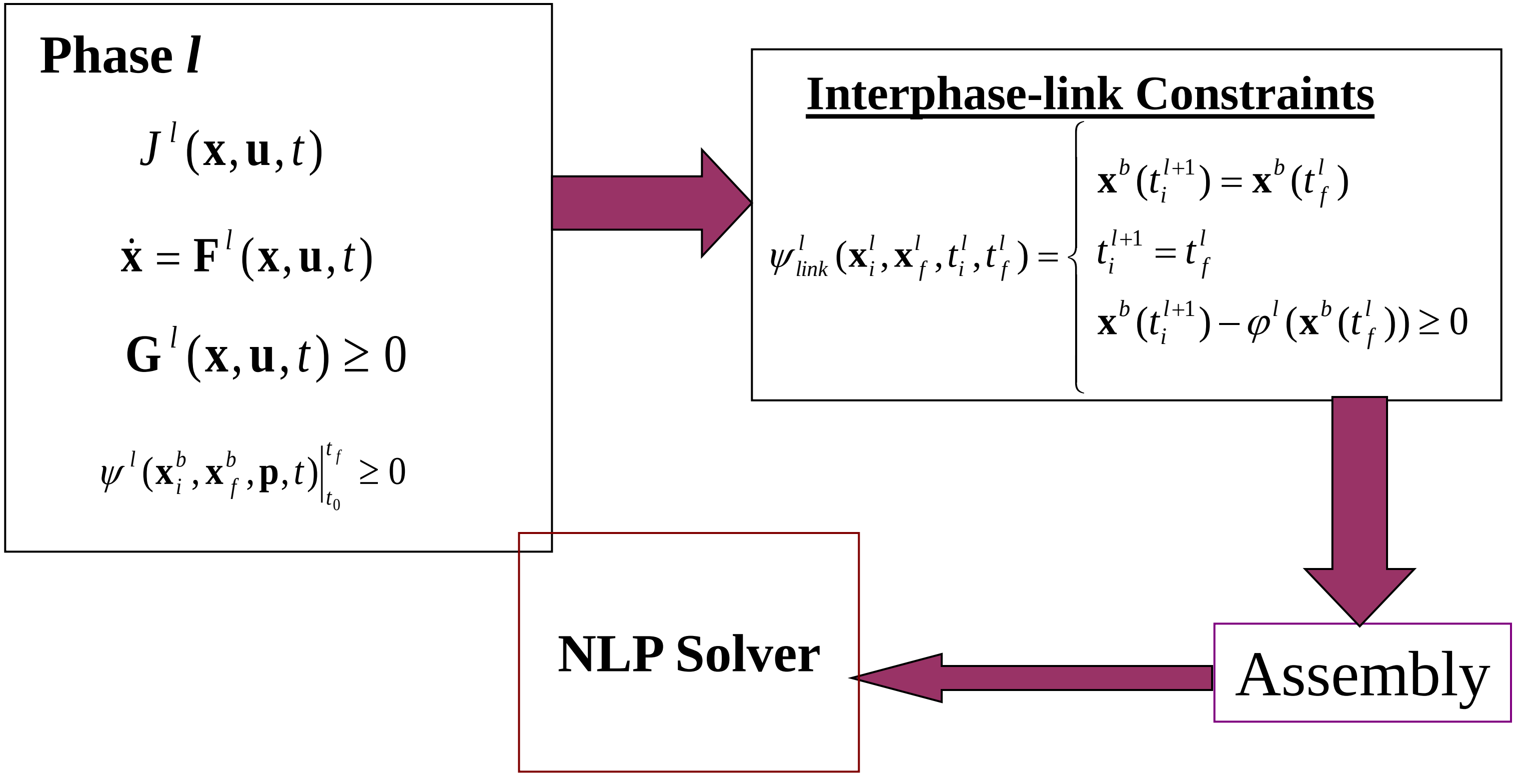
where

$$\mathbf{y}=[\mathbf{x},\mathbf{u},t_i,t_f]^{\mathrm{T}}$$

$$\mathbf{c}^l(\mathbf{y}) = \begin{cases} \sum_{i=1}^{q} \sigma_i \left[\dot{\mathbf{w}}_k(\tau_i)^T \mathbf{x}_s(\tau_i) + \mathbf{w}_k(\tau_i)^T \mathbf{F}_s(\tau_i)\frac{\Delta t_j}{2}\right] - \mathbf{w}_{p+1}^T\mathbf{x}_j^b + \mathbf{w}_1^T \mathbf{x}_{j-1}^b = 0 \\ \mathbf{G}_s(\mathbf{x}_s(\tau_i),\mathbf{u}_s(\tau_i),\tau_i) \ge 0 \\ \vdots \\ \phi^l \end{cases}$$

Multi-Phase DFET

- Discontinuous states can be inserted at the boundary of two phases.
- The end states of one phase are matched to the beginning states of another phase through some interphase constraints or...
- ...phases can be in parallel.



The KKT Optimality Conditions

- Lagrangian function of the NLP problem:

$$
\begin{aligned}
\mathfrak{L}(\mathbf{x}_s, \mathbf{u}_s, \varpi, \nu, \eta, t_0, t_f) = {} & J(\mathbf{x}_s, \mathbf{u}_s, t_0, t_f, \mathbf{X}_0^b, \mathbf{X}_f^b) - \sum_{k=0}^{p+1} \varpi_k \sum_{i=1}^{q} \sigma_i \left[g_k(\tau_i) \mathbf{F}(\mathbf{X}(\tau_i), \mathbf{U}(\tau_i), \tau_i) \Delta t/2 + \dot{g}_k(\tau_i) \mathbf{X}(\tau_i) \right] \\
& - \sum_{s=0}^{p} \eta_s \mathbf{G}(\mathbf{x}_s, \mathbf{u}_s, t_s) - \varpi_1 \mathbf{X}_i^b + \varpi_{p+1} \mathbf{X}_f^b - \varrho \Psi(\mathbf{X}_0^b, \mathbf{X}_f^b, t_0, t_f)
\end{aligned}
$$

- According to Karush-Kuhn-Tucker first order optimality conditions:

$$\frac{\partial \mathfrak{L}}{\partial u_s} = 0 \qquad \frac{\partial \mathfrak{L}}{\partial x_s} = 0$$

$$\frac{\partial \mathfrak{L}}{\partial \varpi_k} = 0 \qquad \frac{\partial \mathfrak{L}}{\partial t_f} = 0$$

$$\frac{\partial \mathfrak{L}}{\partial \varpi} = 0 \qquad \frac{\partial \mathfrak{L}}{\partial \nu} = 0$$

$$\frac{\partial \mathfrak{L}}{\partial X_f} = 0 \qquad \frac{\partial \mathfrak{L}}{\partial X_0} = 0$$

- With transversality conditions:

$$\frac{\partial \mathfrak{L}}{\partial X_i^b} = \frac{\partial \phi}{\partial X_i^b} - \varpi_0 - \varrho \frac{\partial \Psi}{\partial X_i^b} = 0$$

$$\frac{\partial \mathfrak{L}}{\partial X_f^b} = \frac{\partial \phi}{\partial X_f^b} + \varpi_{p+1} - \varrho \frac{\partial \Psi}{\partial X_f^b} = 0$$

- Which are equivalent to the maximum principle transversality conditions.
- KKT Optimality conditions on controls and costates:

$$\sum_{i=1}^{q} \frac{\partial L(\tau_i)}{\partial u} \frac{\Delta t}{2} f_s(\tau_i) - \sum_{i=1}^{q} \sigma_i \sum_{k=0}^{p+1} \varpi_k g_k(\tau_i) \frac{\partial \mathbf{F}(\tau_i)}{\partial u} \frac{\Delta t}{2} f_s(\tau_i) - \eta_s \frac{\partial G}{\partial u} f_s(\tau_i) = 0$$

$$\sum_{i=1}^{q} \frac{\partial L(\tau_i)}{\partial x} \frac{\Delta t}{2} f_s(\tau_i) - \sum_{i=1}^{q} \sigma_i \Big(\sum_{k=0}^{p+1} \varpi_k \dot{g}_k(\tau_i) + \sum_{k=1}^{p+1} \varpi_k g_k(\tau_i) \frac{\partial \mathbf{F}(\tau_i)}{\partial x} \frac{\Delta t}{2} \Big) f_s(\tau_i) - \eta_s \frac{\partial G}{\partial x} f_s(\tau_i) = 0$$

- If one assumes that:

$$\eta_s = \eta(\tau_s)\sigma_s \frac{\Delta t}{2} \qquad \varpi(\tau_i) = \sum_{k=0}^{p+1} \varpi_k g_k(\tau)$$

And

$$\lambda = -\varpi \qquad \mu = -\eta,$$

- Then one gets the integral forms of the optimality conditions:

$$\int_{t_0}^{t_f} \Big(\frac{\partial J}{\partial u} - \varpi \frac{\partial F}{\partial u} - \eta \frac{\partial G}{\partial u} \Big) d\tau = 0$$

$$\int_{t_0}^{t_f} \Big(\frac{\partial J}{\partial x} - \dot{\varpi} - \varpi \frac{\partial F}{\partial x} - \eta \frac{\partial G}{\partial x} \Big) d\tau = 0$$

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Example: Minimum quadratic control with path constraints

- Simple linear system with minimization of the integral of the square of the control action:

$$\min_u J = \frac{1}{2}\int_{t_0}^{t_f} u^2 \, dt$$

- Linear differential constraints

$$\dot{x} = v$$
$$\dot{v} = u$$

- Path constraint:

$$x - l \leq 0$$

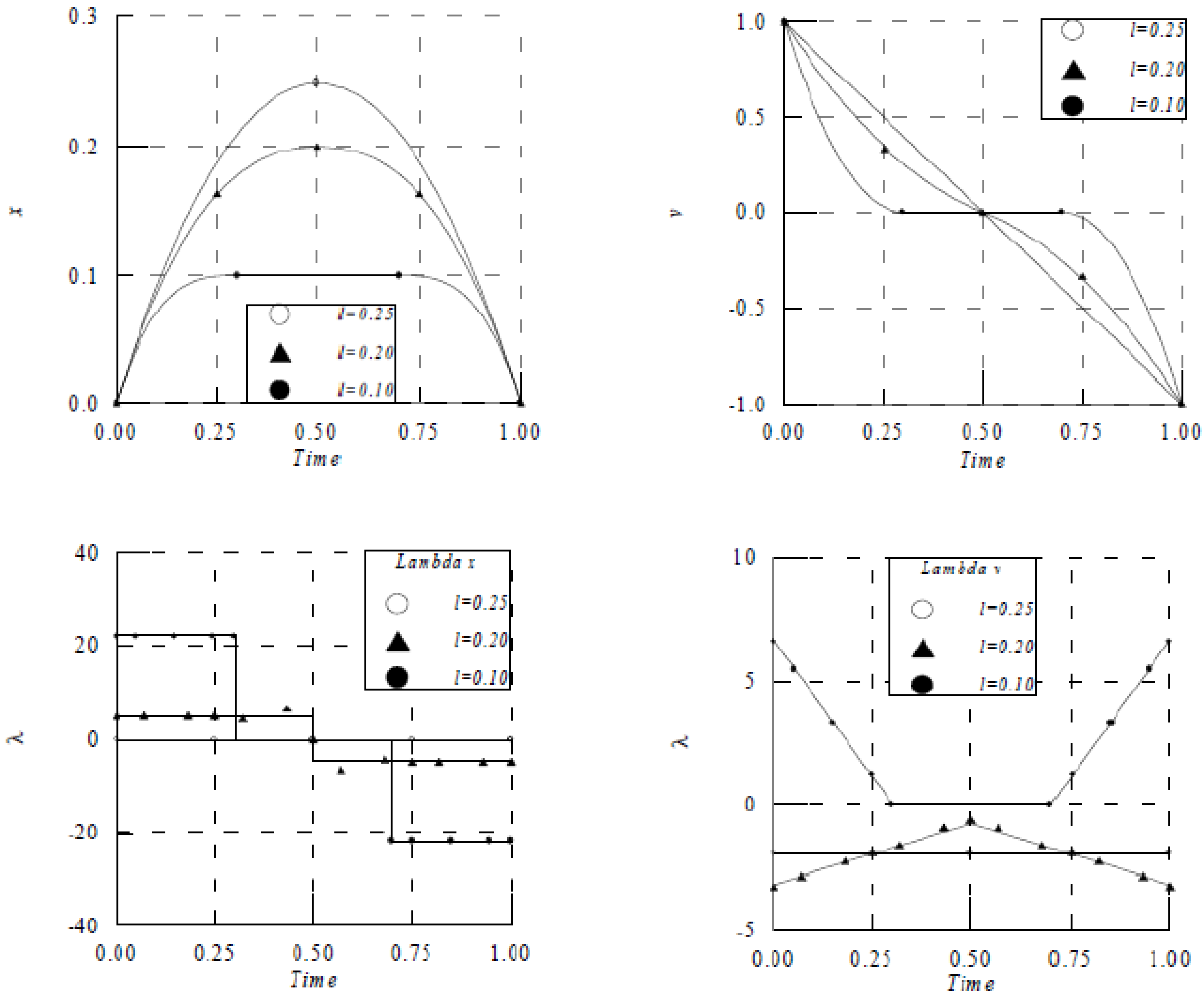
- Boundary conditions:

$$\Psi\big|_{t_0} = \left\{ \begin{array}{c} x \\ v \end{array} \right\}_{t=0} = 0 \qquad \Psi\big|^{t_f} = \left\{ \begin{array}{c} x \\ v+1 \\ t_f - 1 \end{array} \right\}^{t=t_f} = 0;$$

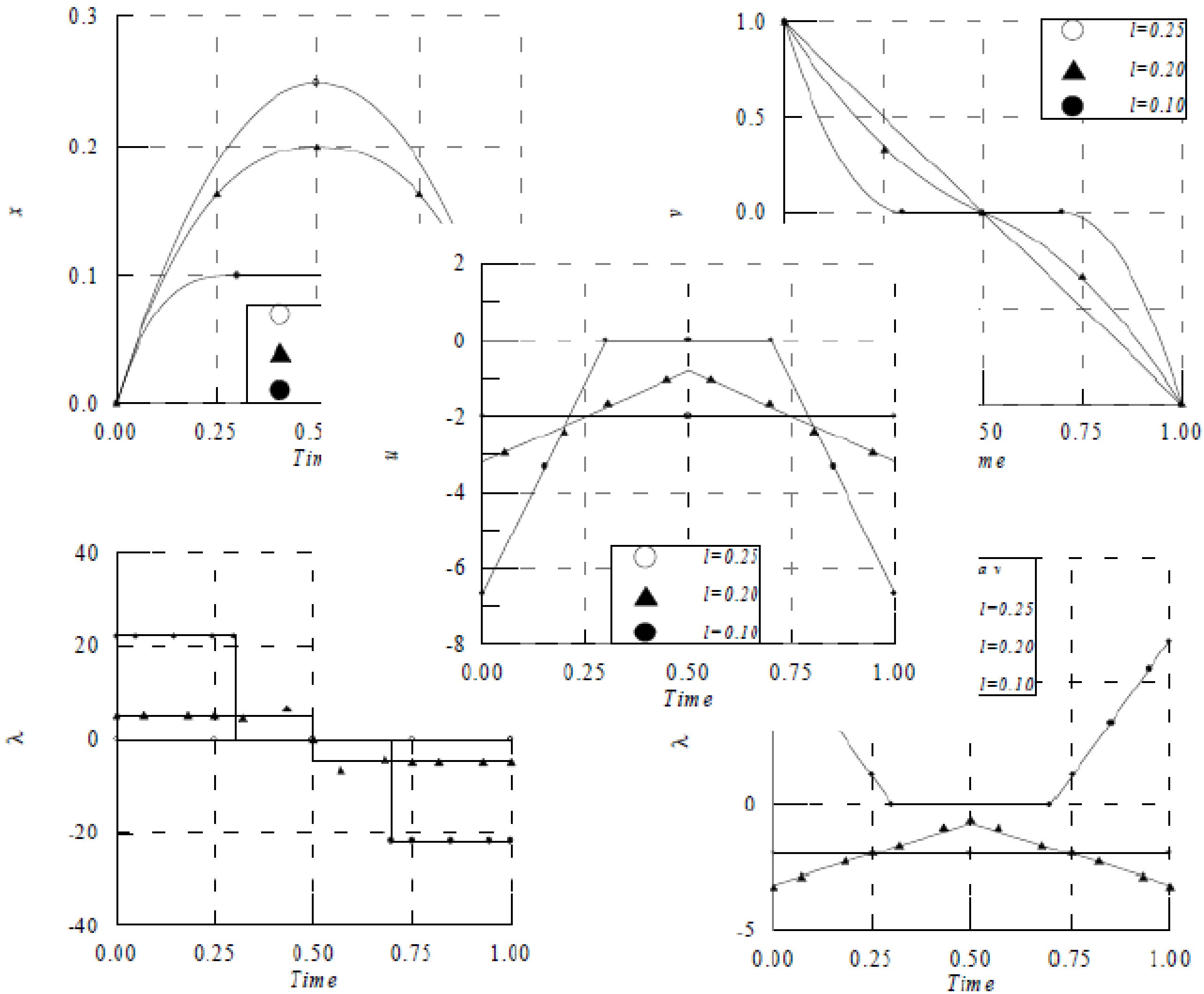
Table 1. Node and Element Distribution

Upper limit		Nodes per Element
l=0.25	states	3:3
	controls	2:2
l=0.20	states	3:3:3:3
	controls	2:2:2:2
l=0.10	states	4:1:4
	controls	3:1:3

Example: Minimum quadratic control with path constraints



Example: Minimum quadratic control with path constraints



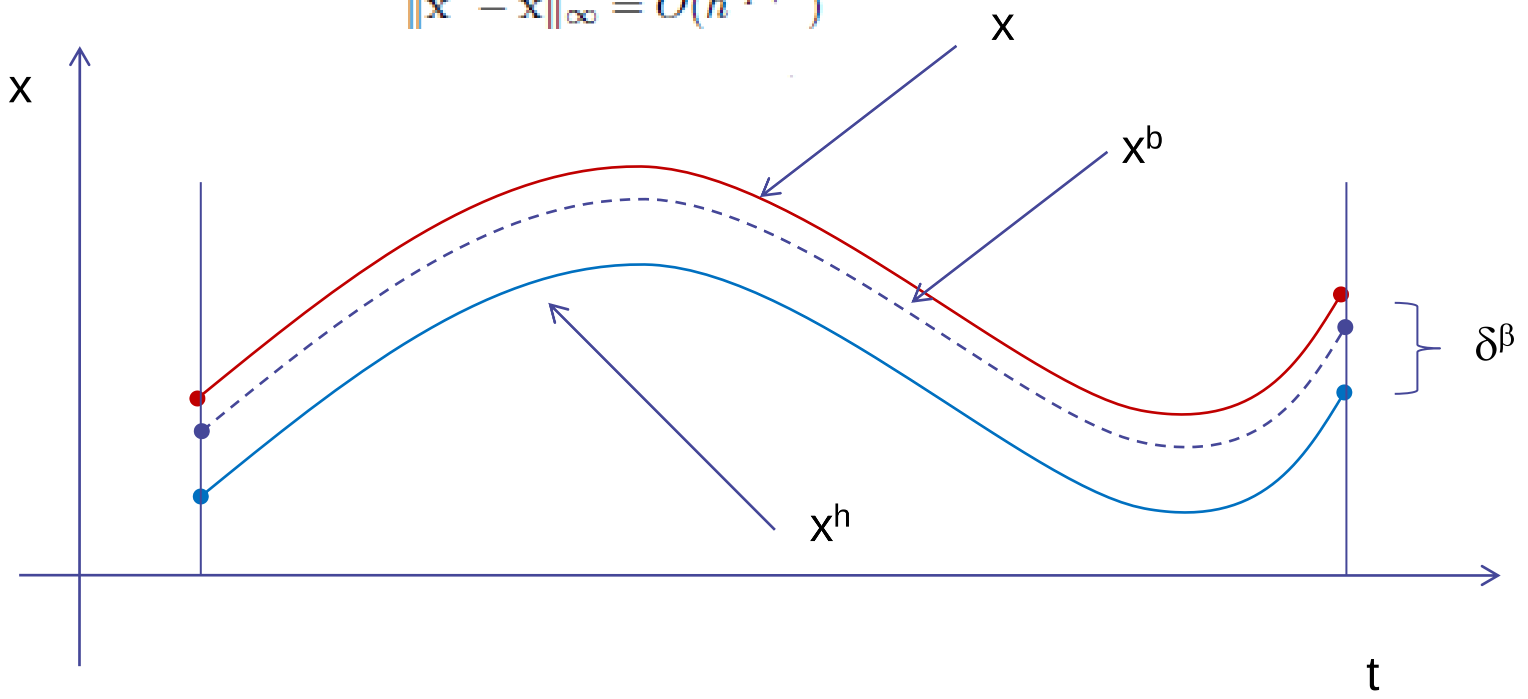
Discontinuity at the Boundaries

- From the theory of Delfour et al. 1981 on Galerkin methods for ODEs:

$$|(\mathbf{x}^b - \mathbf{x})|_\infty \le ch^p \|(\mathbf{x}^h - \mathbf{x})\|$$

- And a maximum convergence rate of:

$$\|\mathbf{x}^b - \mathbf{x}\|_\infty = O(h^{2p+2})$$



- Delfour also demonstrated the accuracy of FET developed on Gauss Legendre, Radau and Lobatto points.

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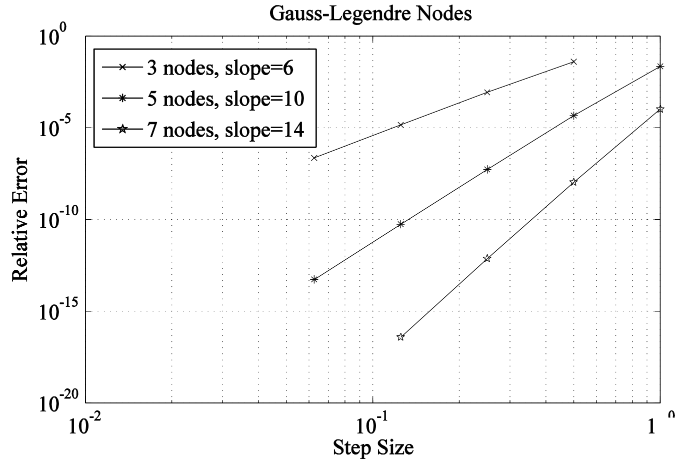
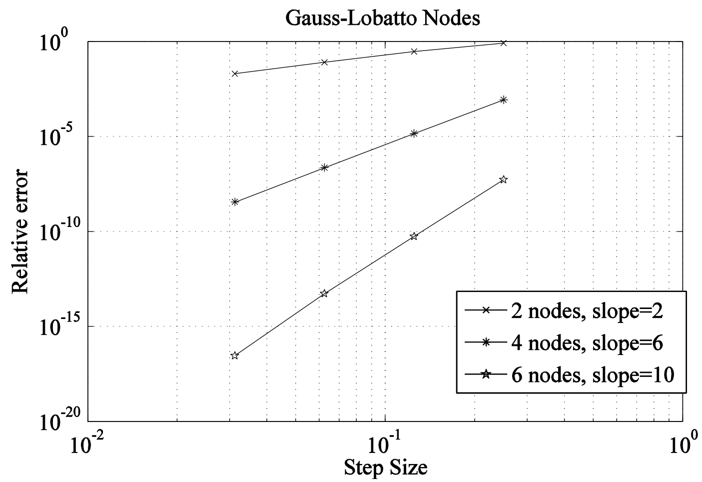
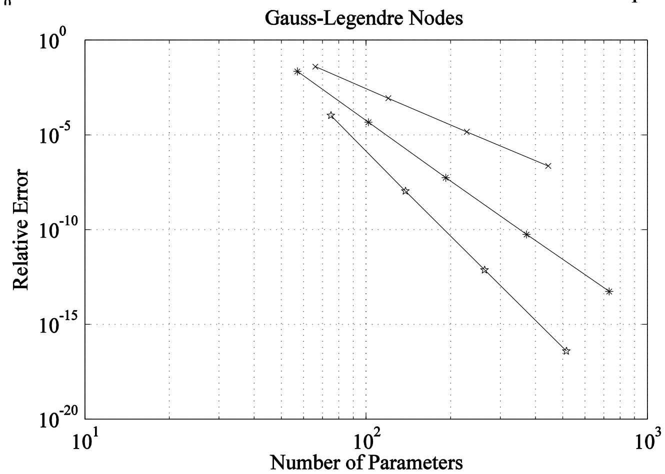




- Simple double integrator:



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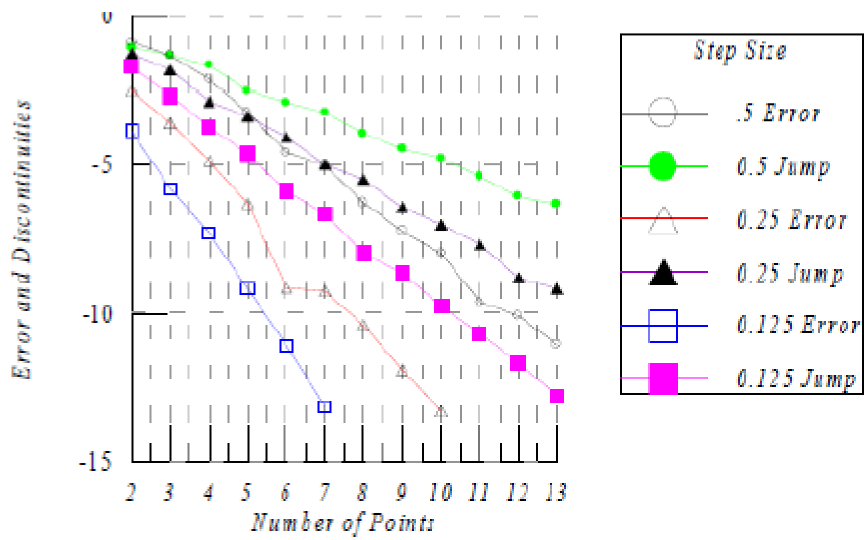
Discontinuity at the Boundaries

- The expected convergence of the discontinuity at the boundary is O(h^{p+2}):

$$\|\mathbf{e}^b\|_\infty = \|\mathbf{x}^b - \mathbf{x}\|_\infty \leq c h^p c_2 |\delta^b|_\infty$$

- Example

Elliptical orbit motion:



- Therefore one could use the following indicator $\epsilon_p^j = \dfrac{|\delta^b|_\infty}{\max|\mathbf{x}^b| + 1} = c_p^j h_j^{p+2}$

$$c_p^j = \frac{\epsilon_p^j}{h_j^{p+2}}$$

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h-p Adaptivity Strategies

- p-adaptivity $\epsilon_p^j = c_p^j h_j^{p+2}$ $\epsilon_\nu^j = tol = c_p^j h_j^{\nu_j}$

$$\nu_j = \frac{(p+2)\log h_j - \log(\frac{\epsilon_p^j}{tol})}{\log h_j}$$

- h-adaptivity
- Given the residual on the error at each element: $r_h^j = tol - c_p^j h_j^{\nu_j}$
- Solve the following linear quadratic optimization problem:

$$\min_{h_j} \sum_{j=1}^{N} (r_\eta^j)^2$$

$$\sum_{j=1}^{N} h_j = 1$$

$$h_j \geq \varepsilon$$

If the residuals are not zero, then all the elements with a nonzero residual are split in two.

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Examples: Minimum Time to Orbit

- The problem is to minimize the time to reach a given altitude with a given velocity (Bryson and Ho 1979):

$$\min_{\beta} t_f$$

- With the dynamic constraints:

$$\dot{x} = u$$
$$\dot{y} = v$$
$$\dot{u} = a\cos\beta(t)$$
$$\dot{v} = -g + a\sin\beta(t)$$

- And the boundary conditions:

$$\Psi\big|_{t_0} = \left\{\begin{matrix} x \\ y \\ u \\ v \end{matrix}\right\}_{t=0} = 0 \qquad \Psi\big|^{t_f} = \left\{\begin{matrix} y-h \\ u-U \\ v \end{matrix}\right\}^{t=f} = 0$$

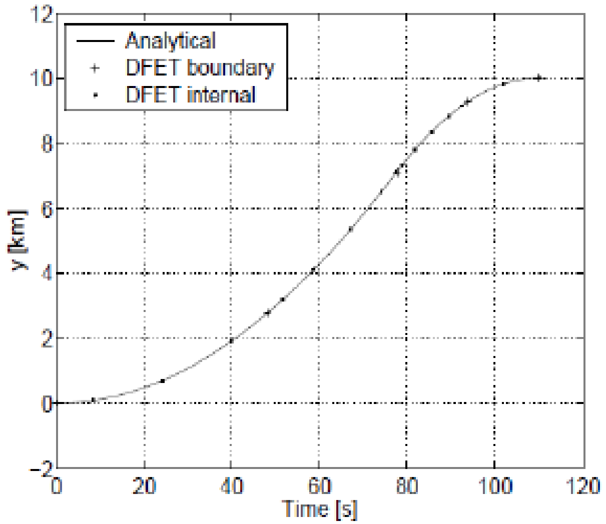
- From Pontryagin maximum principle one gets the control law:

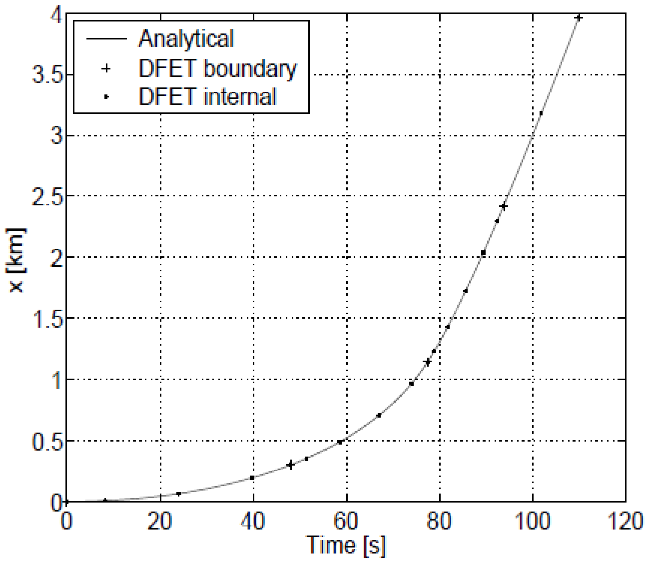
$$\tan\beta = \frac{\lambda_v}{\lambda_u}$$

Examples: Minimum Time to Orbit

Table 2. Optimal Final Time for Different Mesh Grids

Mesh Grid	NLP Parameters	Feasibility	Optimality
2:2:2:2	48	8.54166e-2	1.679041e-2
3:3:3:3	68	3.73264e-2	2.118610e-3
4:4:4:4	88	1.28994e-2	3.854148e-4
5:5:5:5	108	9.57079e-3	4.162192e-5
6:6:6:6	128	1.37934e-3	7.136340e-5
7:7:7:7	148	1.70779e-3	8.372763e-6
Adapted	118	8.26464e-4	8.363671e-6





Examples: Minimum Time to Orbit

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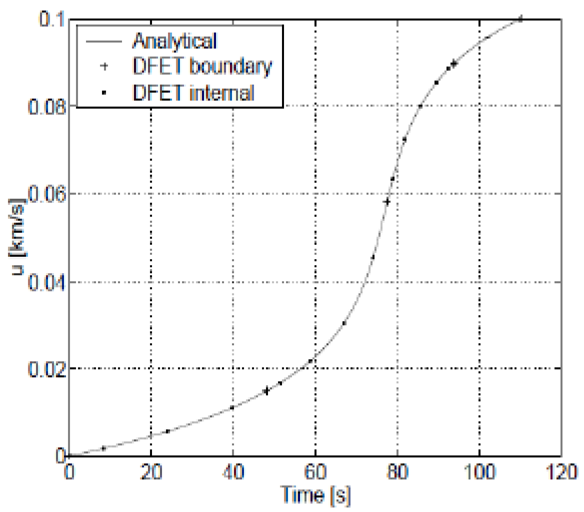
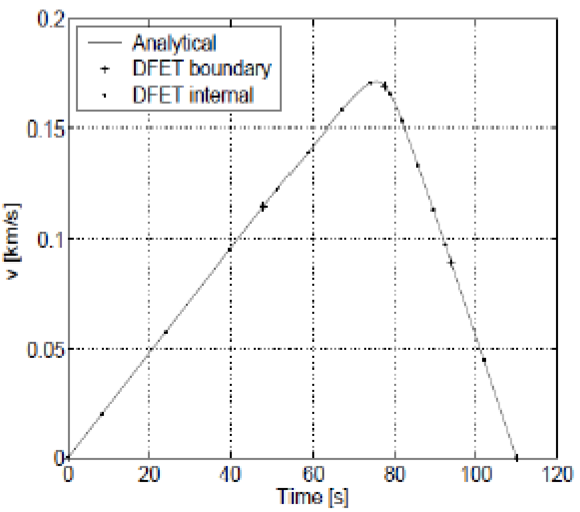
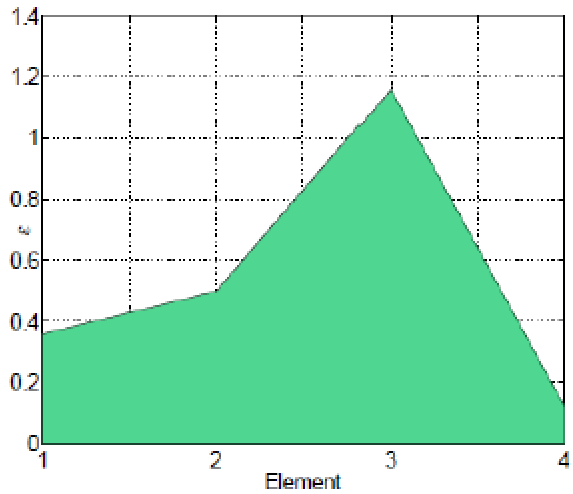
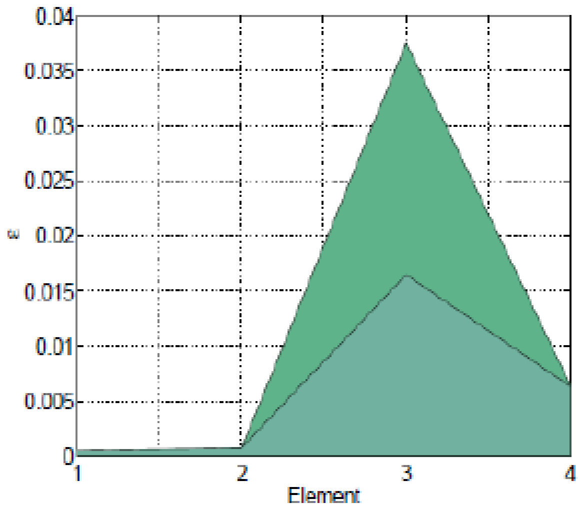
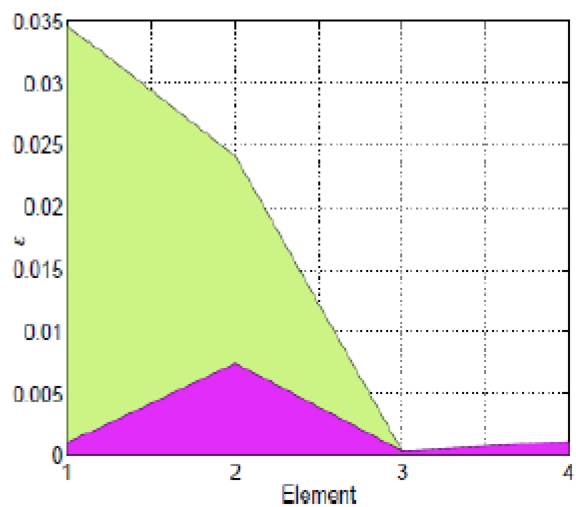


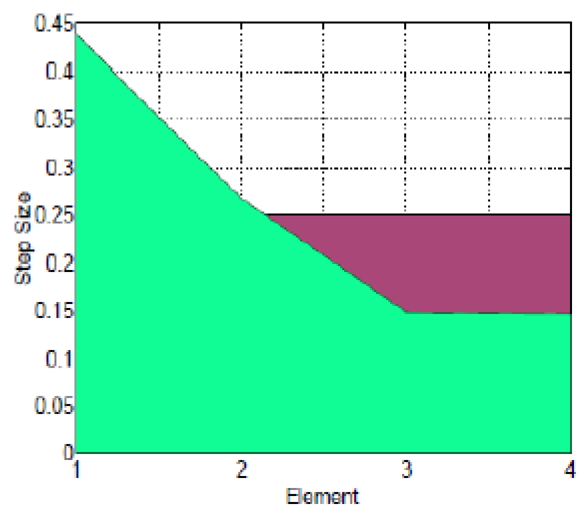


Examples: Step-by-Step Adaptation Process

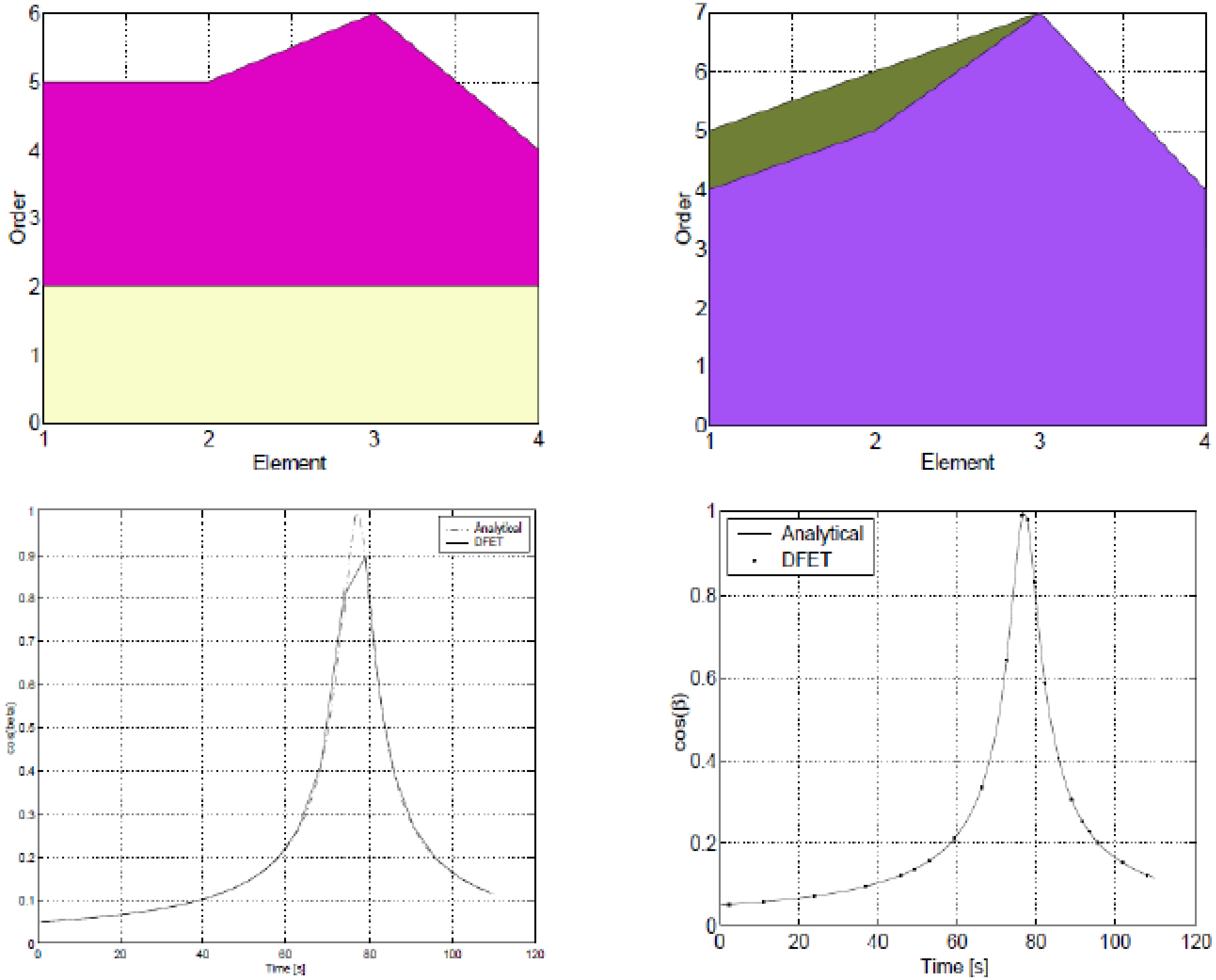








Examples: Step-by-Step Adaptation Process



Robust Control

- Given a solution of the NLP problem, one has **C**(**x**)=0.
- By linearizing in a neighbourhood of **x** one can get:

$$\begin{bmatrix} \nabla C_{U_s} & \nabla C_{X_i} & \nabla C_{X_s} & \nabla C_{X_f} \\ 0 & \nabla C_{B_i} & 0 & \nabla C_{B_f} \\ \nabla G_{U_s} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{U}_s \\ \Delta \mathbf{X}_i \\ \Delta \mathbf{X}_s \\ \Delta \mathbf{X}_f \end{Bmatrix} = 0$$

- Where the matrix is the Jacobian of the constraints whose components are the gradients with respect to states and controls.
- If the uncertainty on the initial state is given and no variation on the final state is required, one can solve the following linear quadratic optimisation problem:

$$\min_{\Delta \mathbf{U}} \frac{1}{2} \Delta \mathbf{U}^T \Delta \mathbf{U}$$

$$s.t.$$

$$\mathbf{J}_{11} \Delta \mathbf{U}_s + \mathbf{J}_{12} \Delta \mathbf{X}_i + \mathbf{J}_{13} \Delta \mathbf{X}_s = 0$$

$$\Delta \mathbf{X}_i = \Xi; \quad \Delta \mathbf{X}_f = 0$$

Transition Matrix

- If the control is considered to be fixed, the Jacobian matrix can be used to derive the state transition matrix from initial states to final states:

$$\begin{bmatrix} \mathbf{S}_i & 0 & 0 \\ \mathbf{S}_{io} & \mathbf{S}_{oi} & 0 \\ 0 & \mathbf{S}_{of} & \mathbf{S}_{fo} \\ 0 & 0 & \mathbf{S}_f \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{X}_{s_i} \\ \Delta\mathbf{X}_{s_o} \\ \Delta\mathbf{X}_{s_f} \end{Bmatrix} = \begin{bmatrix} \Delta\mathbf{X}_i \\ 0 \\ 0 \\ \Delta\mathbf{X}_f \end{bmatrix}$$

- Starting from the bottom of the matrix one can compute:

$$\Delta X_f = S_f \Delta X_{s_f}$$

- And then proceed with::

$$\Delta X_{s_f} = -S_{fo}^{-1} S_{of} \Delta X_{so}$$

$$\Delta X_{s_o} = -S_{oi}^{-1} S_{io} \Delta X_{si}$$

- Up to:

$$\Delta X_{s_i} = -S_i^{-1} \Delta X_i$$

- with

$$\Delta\mathbf{X}_f = \mathfrak{T} \Delta\mathbf{X}_i.$$

$$\mathfrak{T} = -S_f S_{fo}^{-1} S_{of} S_{oi}^{-1} S_{io} S_i^{-1}$$

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GAUSS PSEUDO-SPECTRAL METHODS (GPSM)

Pseudo-spectral Transcription

Consider again the differential equations:

$$\dot{\mathbf{x}} - \mathbf{F}(\mathbf{x}, \mathbf{u}, t) = 0$$

And the polynomial representation of states and controls:

$$\begin{Bmatrix} \mathbf{x} \\ \mathbf{u} \end{Bmatrix} = \sum_{s=1}^{p} f_s(t) \begin{Bmatrix} \mathbf{x}_s \\ \mathbf{u}_s \end{Bmatrix}$$

With the first derivatives:

$$\mathbf{x} = \sum_{s=1}^{p} \dot{f}_s(t)\mathbf{x}_s$$

Let's assume now that the nodes $\mathbf{x}_s$ are chosen to be the zeros τ_s of orthogonal polynomial, like Legendre polynomials, in the interval [-1,1].

Then *f(t)* is different from zero at each $t=\tau_s$.

Pseudo-spectral Transcription

Then we can collocate the differential equations as follows:

$$\sum_{k=0}^{M} D_{s,k}\mathbf{x}_k - \mathbf{F}(\mathbf{x}_s, \mathbf{u}_s, \tau_s)\frac{\Delta t}{2} = 0; \quad \mathrm{s} = 1, \ldots, M$$

Now consider the stong integral form:

$$\mathbf{x}(t_f) = \mathbf{x}(t_0) + \int_{t_0}^{t_f} \mathbf{F}(\mathbf{x}, \mathbf{u}, t)dt$$

And assume the integral term is integrated numerically with a Gauss quadrature formula:

$$\mathbf{x}(t_f) = \mathbf{x}(t_0) + \sum_{s=1}^{M} \sigma_s \mathbf{F}(\mathbf{x}_s, \mathbf{u}_s, \tau_s)\frac{\Delta t}{2}$$

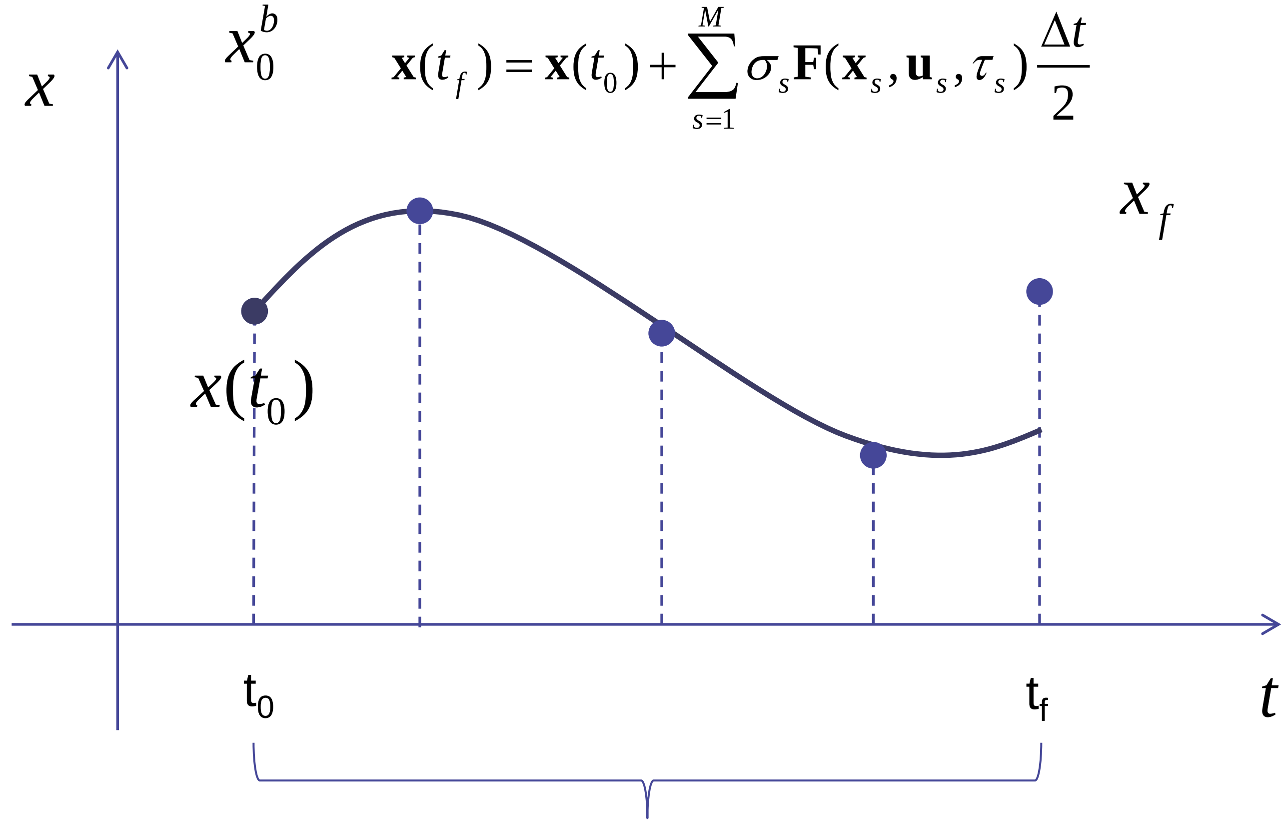
Then one can choose a Lagrange interpolating polynomial that interpolates $\mathbf{x}_0$ plus the *M* Gauss points $\mathbf{x}_s$ to represent the states (the controls repsectively).

Pseudo-spectral Transcription

The complete set of equations becomes:

$$\sum_{k=0}^{M} D_{s,k}\mathbf{x}_k - \mathbf{F}(\mathbf{x}_s, \mathbf{u}_s, \tau_s)\frac{\Delta t}{2} = 0; \quad \mathrm{s} = 1, \ldots, M$$

$$\mathbf{x}(t_f) = \mathbf{x}(t_0) + \sum_{s=1}^{M} \sigma_s \mathbf{F}(\mathbf{x}_s, \mathbf{u}_s, \tau_s)\frac{\Delta t}{2}$$



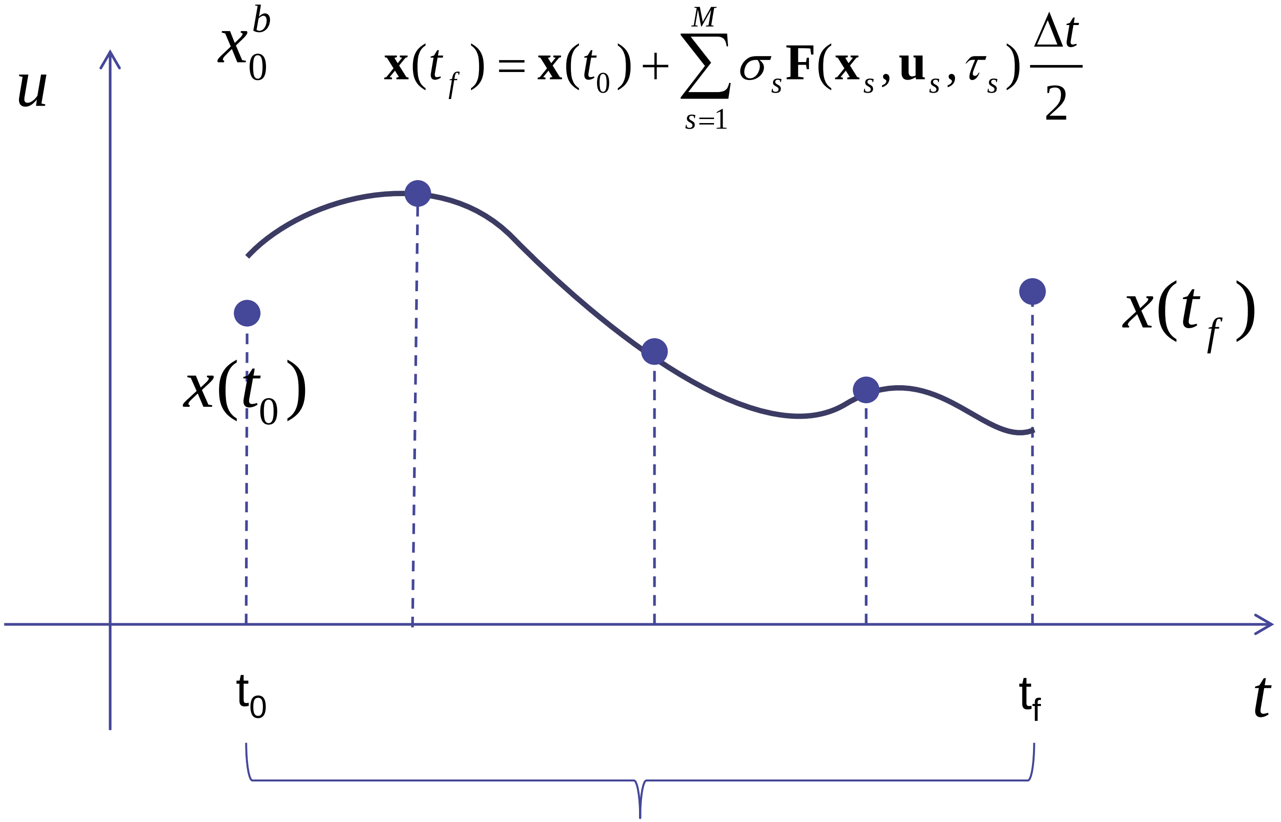
Spectral basis: Gauss points used for integration and discretisation

Pseudo-spectral Transcription

The controls are collocated only at the Gauss points:

$$\sum_{k=0}^{M} D_{s,k}\mathbf{x}_k - \mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,\tau_s)\frac{\Delta t}{2} = 0; \quad \mathrm{s} = 1,\ldots,M$$

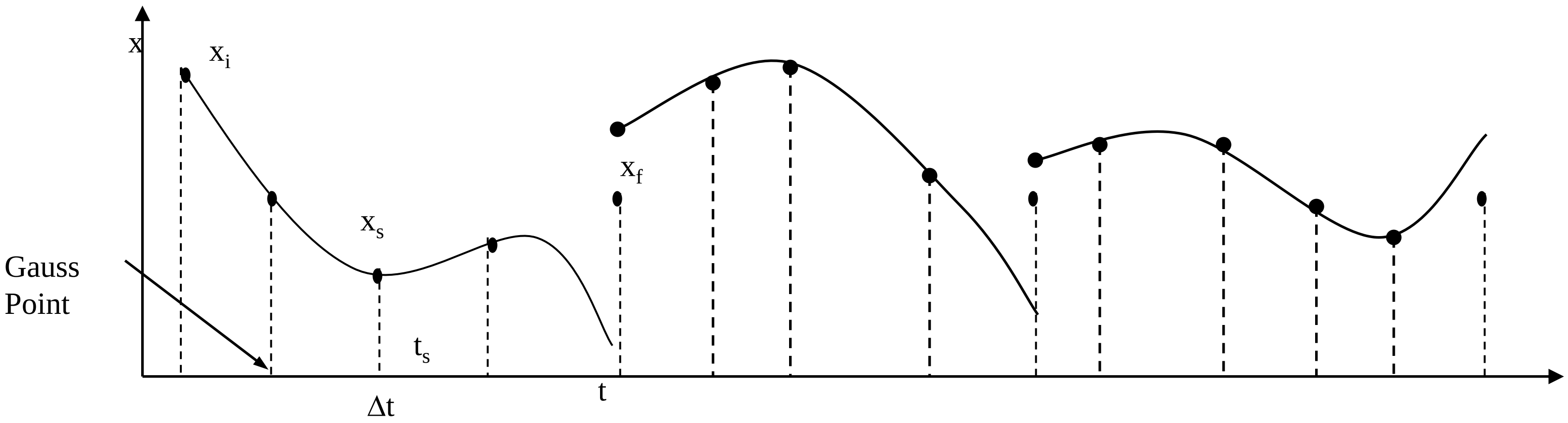
$$\mathbf{x}(t_f) = \mathbf{x}(t_0) + \sum_{s=1}^{M}\sigma_s\mathbf{F}(\mathbf{x}_s,\mathbf{u}_s,\tau_s)\frac{\Delta t}{2}$$



Spectral basis: Gauss points used for integration and discretisation

Multiple Phases Pseudo-spectral Transcription

As for DFET one can partition the time domain in segments and use a PS transcription on each segment:



It is now clear that in order to restore the continuity from one segment to the next one has to add a matching condition:

$$\mathbf{x}_{j+1}(t_{f,j+1}) = \mathbf{x}_j(t_{0,j}); \quad j = 1, \ldots, n_{segment} - 1$$

$$t_{f,j+1} = t_{0,j}$$

The matching condition restores the continuity on the states but not on the control that can remain discontinuous.

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BASICS OF NONLINEAR PROGRAMMING

Simple 1D Newton Method

- Find x^* solution of the nonlinear problem $c(x^*)=0$
- Given a first guess x and a linear expansion of c in x

$$c(\bar{x}) \cong c(x) + \frac{dc(x)}{dx}(\bar{x} - x)$$

The value of x is updated with the Newton's step p given by

$$\bar{x} = x - \left(\frac{dc(x)}{dx}\right)^{-1} c(x) = x + p$$

- Quadratic convergence if $c \in C^1$ and $c_{xx} \neq 0$
- Problem: computing the derivative of c

Simple 1D Minimisation

- Find *x* that minimises *F(x)*
- Given a value *x*, the function *F* is expanded up to the second order:

$$F(\bar{x}) \cong F(x) + F'(x)(\bar{x} - x) + \frac{1}{2}(\bar{x} - x)F''(x)(\bar{x} - x)$$

First order necessary condition is to have:

$$\frac{dF(\bar{x})}{dx} = F'(\bar{x}) = 0 = F'(x) + F''(x)(\bar{x} - x)$$

Sufficient condition for optimality

$$F''(\bar{x}) > 0$$

Newton's method in N dimensions

- Find the vector $\mathbf{x}=[x_1,\ldots,x_n]^T$ such that:

$$\mathbf{c}(\mathbf{x}) = \begin{bmatrix} c_1(\mathbf{x}) \\ \vdots \\ c_n(\mathbf{x}) \end{bmatrix} = \mathbf{0}$$

$$\bar{\mathbf{x}} = \mathbf{x} + \mathbf{p}$$

$$\mathbf{G}\mathbf{p} = -\mathbf{c}$$

where **G** is the Jacobian matrix of nonlinear functions **c**

Newton's method for minimisation in N dimensions

- Find $\mathbf{x}=[x_1,\ldots,x_n]^T$ that minimises $F(\mathbf{x})$:

First order necessary conditions for optimality

$$g(\mathbf{x}) = \begin{bmatrix} g_1(\mathbf{x}) = \dfrac{dF}{dx_1} \\ \vdots \\ g_n(\mathbf{x}) = \dfrac{dF}{dx_n} \end{bmatrix} = \mathbf{0}$$

$$\bar{\mathbf{x}} = \mathbf{x} + \mathbf{p}$$

$$\mathbf{H}\mathbf{p} = -\mathbf{g}$$

Where **H** is the Hessiana matrix of *F*

Approximation of the Hessian and Jacobian matrix

- Build an estimate B* of matrix B:

$$\mathbf{B}^* = \mathbf{B} + h(\Delta c, \Delta x)$$

$$h = \frac{(\Delta c - \mathbf{B}\Delta x)\Delta x^T}{\Delta x^T \Delta x}$$

Update the Hessian matrix with the BFGS (Broyden-Fletcher-Goldfarb-Shanno) formula:

$$h = \frac{\mathbf{B}\Delta x \Delta x^T \mathbf{B}}{\Delta x^T \mathbf{B} \Delta x}$$

In this case the convergence is no more quadratic but superlinear

Constrained optimisation: Equality Constraints

- find x that minimises F(**x**) subject to:

$$\mathbf{c}(\mathbf{x}) = \mathbf{0}$$

A scalar function called Lagrangian is defined as:

$$L(\mathbf{x}, \lambda) = F(\mathbf{x}) - \lambda^T \mathbf{c}(\mathbf{x}) = F(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i c_i$$

First order necessari conditions for optimality (Kuhn-Tuker):

$$\nabla_x L = \mathbf{g} - \mathbf{G}^T \lambda = \nabla F - \sum_{i=1}^{m} \lambda_i \nabla c_i = 0$$

$$\nabla_\lambda L = -\mathbf{c}(\mathbf{x}) = 0$$

Suffcicient condition for optimality is that the Hessian of the L is:

$$\mathbf{v}^T \mathbf{H}_L \mathbf{v} > 0$$

Constrained optimisation: Equality Constraints

Minimise:

$$F(x) = x_1^2 + x_2^2$$

Subject to

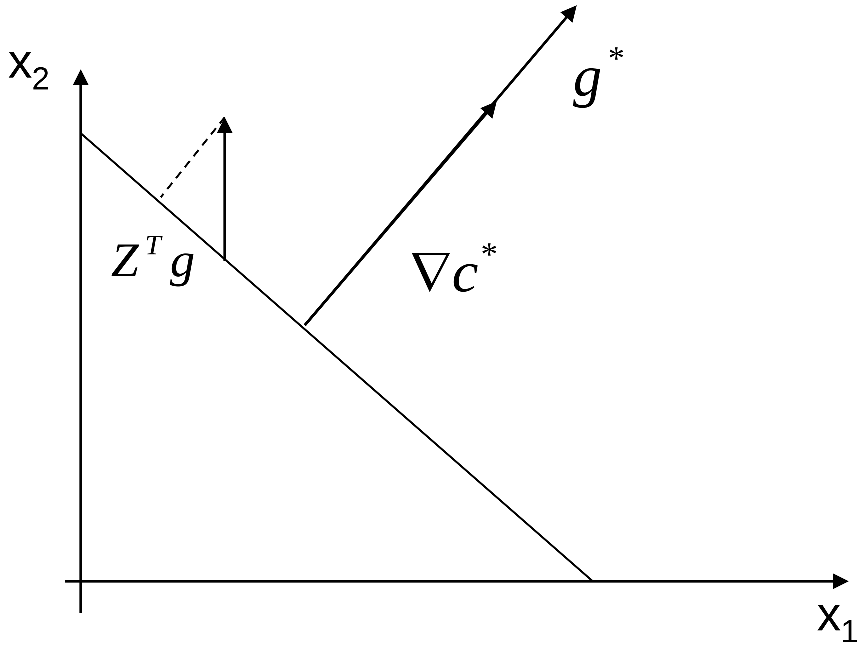
$$c(x) = x_1 + x_2 - 2 = 0$$

And solution

$$x^* = (1,1)$$

At the solution point

$$\mathbf{g} - \mathbf{G}^T \lambda = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} 2 = \mathbf{0}$$



Equivalent necessary condition is that the projection of the gradient of the function *F* on the constraint is zero: $\mathbf{Z}^T \mathbf{g} = \mathbf{0}$

Equivalent sufficient condition is that the projection of the Hessiana is positive defined:

$$\mathbf{Z}^T \mathbf{H}_L \mathbf{Z}$$

Constrained optimisation: Inequality Constraints

- Find x that minimises *F*(**x**) subject to:

$$\mathbf{c}(\mathbf{x}) \geq 0$$

For **x**=**x***:

- Some constraints are equal to zero

$$c_i(\mathbf{x}) = 0 \ \text{ for } i \in A \text{ i.e. belongs to the active set}$$

- Others are strictly satisfied

$$c_i(\mathbf{x}) > 0 \ \text{ for } i \in A' \text{ i.e. belong to the set of inactive constraints}$$

Necessari condition for optimality is that lagrangian multipliers at the solution are:

$$\lambda_i \geq 0$$

Constrained optimisation: Inequality Constraints

- **Two different approaches:**
  - **active set method**
  - **Interior point method**

Active Set

This methods looks for the active set at every optimisation step

Interior Point

Inequality constraints are penalised in the objective function through a barrier function dependent on a penality parameter μ:

$$F(\mathbf{x}) + \mu \ln c(\mathbf{x})$$

The solution is forced to stay within the feasible region by the barrier function *ln*c(**x**)

Constrained optimisation: The K-T system

The solution of the constrained optimisation problem satisfies the following first order conditions:

$$L(\mathbf{x},\lambda,\mu) = F(\mathbf{x}) - \lambda^T \mathbf{c}^e(\mathbf{x}) - \mu^T \mathbf{c}^i(\mathbf{x})$$

$$\nabla_x L = \mathbf{g} - \mathbf{G}_e^T \lambda - \mathbf{G}_i^T \mu = 0$$

$$\nabla_\lambda L = -\mathbf{c}_e(\mathbf{x}) = 0$$

$$\nabla_\mu L = -\mathbf{c}_i(\mathbf{x}) = 0$$

$$\mu \geq 0$$

Slack variables are introduced to insert inequality constraints as equality constraints:

$$L(\mathbf{x},\eta,\mathbf{s}) = F(\mathbf{x}) - \eta^T [\mathbf{c}(\mathbf{x}) - \mathbf{s}]$$

$$\mathbf{s} \geq \mathbf{0}$$

$$\nabla_y L = \mathbf{g} - \mathbf{G}^T \eta = 0$$

$$\nabla_\eta L = -\mathbf{c}\ \ (\mathbf{y}) = 0$$

$$\mathbf{b}_u \geq \mathbf{y} = \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} \geq \mathbf{b}_l$$

$$\eta \geq 0$$

Constrained optimisation: The K-T system

•The direction of steepest descent is computed solving the K-T system:

$$\begin{bmatrix} \mathbf{H}_L & \mathbf{G}^T \\ \mathbf{G} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \xi \end{bmatrix} = \begin{bmatrix} -\mathbf{g} + \mathbf{G}^T \eta \\ \mathbf{c} \end{bmatrix}$$

$$\mathbf{p} = [d\mathbf{x}, d\mathbf{s}]^T \qquad d\mathbf{s} = \mathbf{G}^T d\mathbf{x} + (\mathbf{c} - \mathbf{s})$$

•The solution of the K-T system gives the direction of steepest descent and the step. The step length is corrected by a factor α:

$$\begin{bmatrix} \bar{\mathbf{y}} \\ \bar{\eta} \end{bmatrix} = \begin{bmatrix} \mathbf{y} \\ \eta \end{bmatrix} + \alpha \begin{bmatrix} \mathbf{p} \\ \xi \end{bmatrix}$$

•The value α is such that the bounds on slacks **s** and unknowns **x** are not violated

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Given the quadratic objective function

$$\mathbf{g}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x}$$

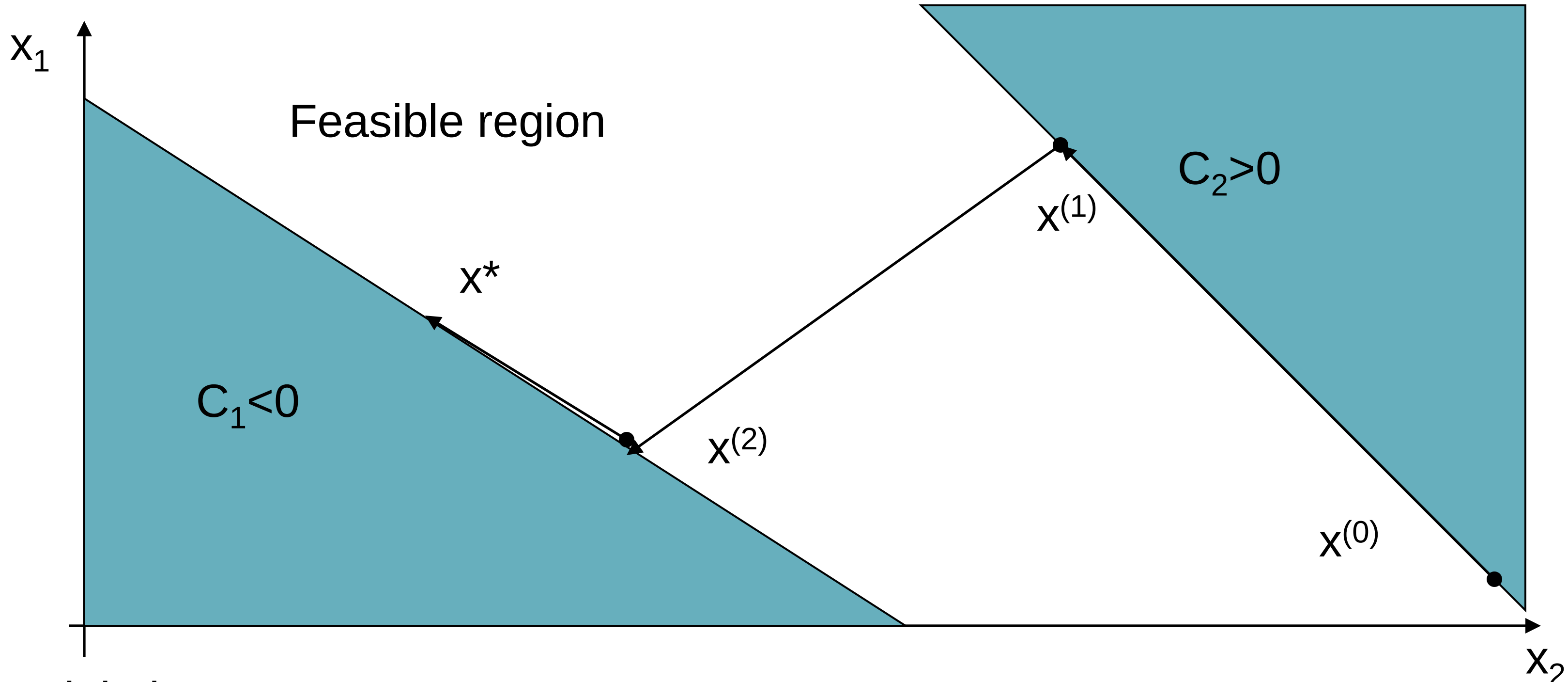
Subject to linear constraints

$$\mathbf{A}\mathbf{x} \geq \mathbf{b}$$

Solution method: starting from a first guess $\mathbf{x}^0$ and an estimate of the active set A^0

1)at step k solve the K-T system with the constraints in A^k as equalities
2)Maximum step length along the direction **p** which does not violate active constraints
3) if the step crosses on inactive constraints the step is reduced and the constraint is inserted among active constraints, then go to step 1 otherwise check the sign of the associated lagrangian multipliers
4) If all the multiplier are positive then stop, otherwise if erase from the active set the constraint associated to the most negative multipliers.

Example



Minimise

$$F(x) = x_1^2 + x_2^2$$

Subject to

$$c_2(x) = 4 - x_1 - \frac{2}{3}x_2 \geq 0$$

$$c_1(x) = x_1 + x_2 - 2 \geq 0$$

Step	x_1	x_2	Active set
0	4	0	$A^0=\{c_2\}$
1	2.77	1.85	λ_2=-5.53 erase c_2
2	1.2	0.8	α=0.5 add c_1
3	1	1	$A^*=\{c_1\}$

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The merit function

Robustness of convergence: local convergence must be acchieved from every starting point in the solution space:

$$M(\mathbf{x},\lambda,\mathbf{s}) = F - \lambda^T(\mathbf{c}-\mathbf{s}) + \frac{1}{2}(\mathbf{c}-\mathbf{s})^T \Pi (\mathbf{c}-\mathbf{s})$$

Betts 2000

where Π is a weight matrix

Unidimensional minimisation in the direction of steepest descent using a quadratic or cubic model modello of the merit function:

$$\begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\lambda} \\ \bar{s} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \lambda \\ s \end{bmatrix} + \alpha \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \\ \sigma \end{bmatrix} \Longrightarrow \quad \min \quad M(\mathbf{x},\lambda,\mathbf{s}) = M(\alpha)$$

NOTICE!
Speed of convergence and robustness improve if Jacobian ed Hessin are analytical

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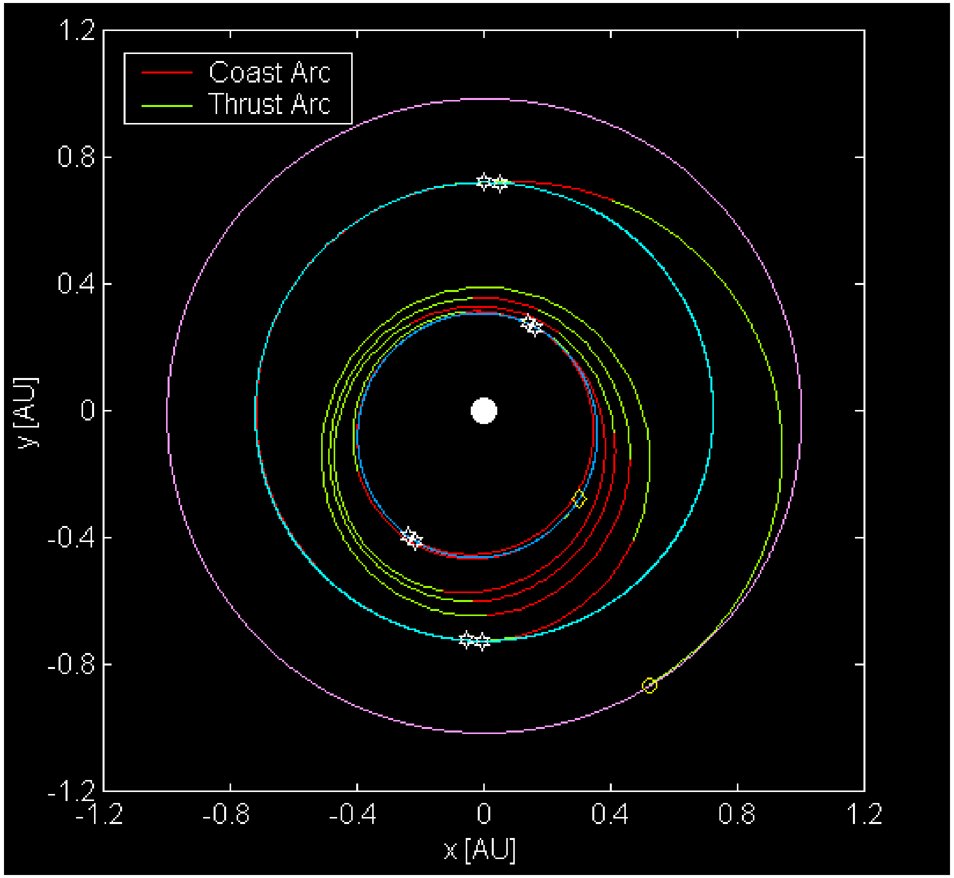
SOME SPACE APPLICATIONS

Direct Finite Element Transcription (DFET)

BepiColombo

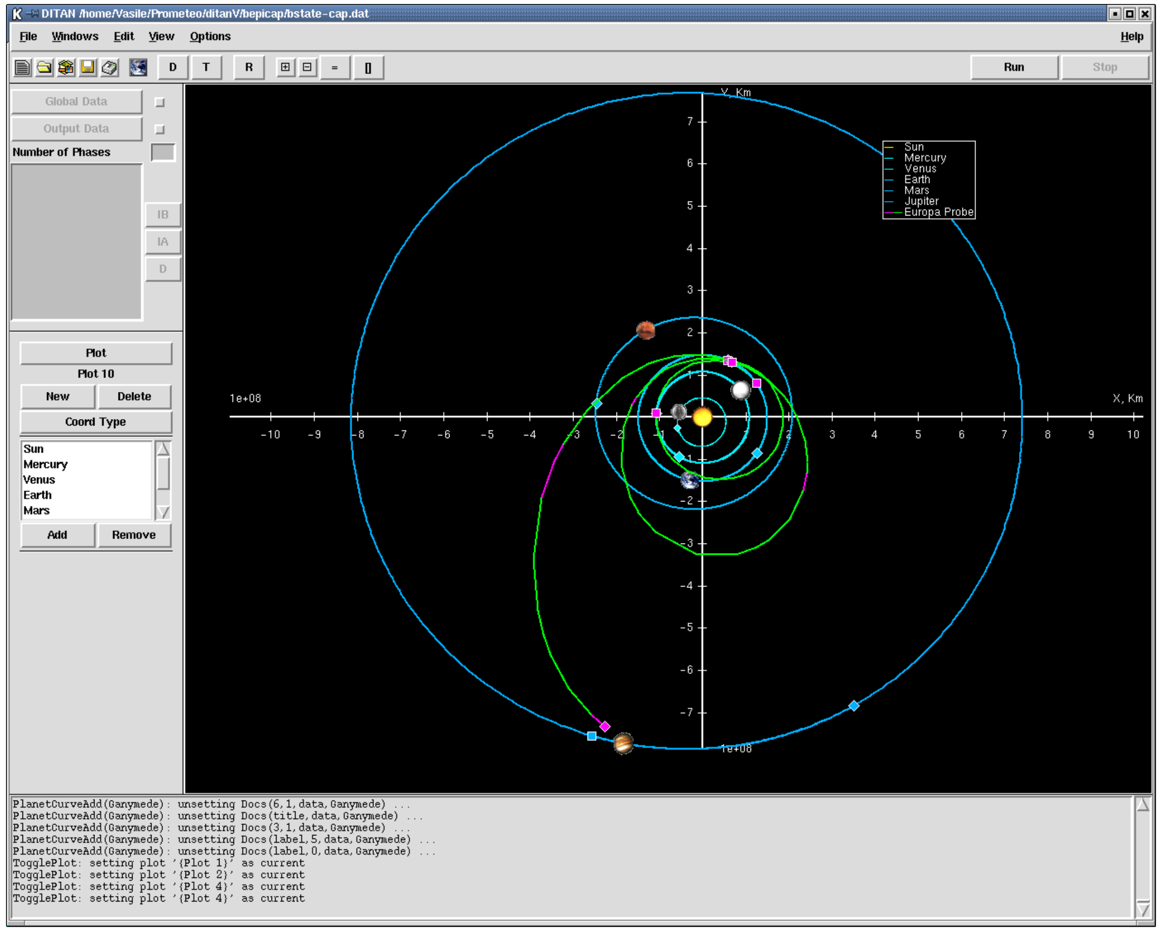
- 3000 variables and constraints for the NLP problem
- 4 to 7 swingbys
- resonant orbits
- more than 20 switching points



Direct Finite Element Transcription (DFET)

Earth-Europa Transfer

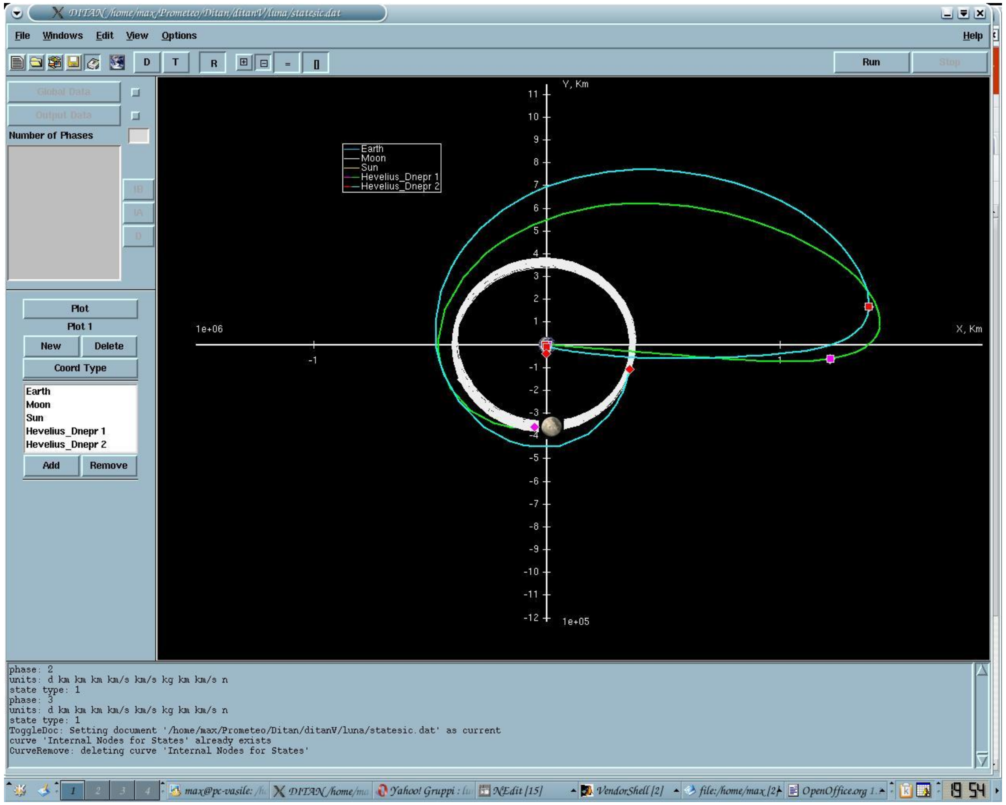
- 6000-7000 variables and constraints for the NLP problem
- 14 swingbys
- resonant orbits
- variable thrust
- Variable reference frames



Direct Finite Element Transcription (DFET)

WSB Transfers

- 1000-1500 variables and constraints
- Highly nonlinear and unstable dynamics
- Impulsive manoeuvres



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FINITE PERTURBATIVE ELEMENTS

Equations of Motion

- Non-singular Equinoctial elements:
  - No singularities for zero-inclination and zero-eccentricity orbits.

$$\mathbf{X} = \begin{Bmatrix} a \\ P_1 = e \cdot \sin(\Omega + \omega) \\ P_2 = e \cdot \cos(\Omega + \omega) \\ Q_1 = \tan\frac{i}{2}\sin\Omega \\ Q_2 = \tan\frac{i}{2}\cos\Omega \\ L = (\Omega + \omega) + \vartheta \end{Bmatrix}$$

- Gauss planetary equations in Equinoctial elements, under a perturbing acceleration ε in the r-t-h frame*:*

$$\frac{da}{dt} = \frac{2a^2}{h}\left[(P_2 \sin L - P_1 \cos L)\varepsilon\cos\beta\cos\alpha + \frac{p}{r}\varepsilon\cos\beta\sin\alpha\right]$$

$$\frac{dP_1}{dt} = \frac{r}{h}\left\{-\frac{p}{r}\cos L \cdot \varepsilon\cos\beta\cos\alpha + \left[P_1 + \left(1 + \frac{p}{r}\right)\sin L\right]\varepsilon\cos\beta\sin\alpha - P_2(Q_1\cos L - Q_2\sin L)\varepsilon\sin\beta\right\}$$

$$\frac{dP_2}{dt} = \frac{r}{h}\left\{-\frac{p}{r}\cos L \cdot \varepsilon\cos\beta\cos\alpha + \left[P_2 + \left(1 + \frac{p}{r}\right)\sin L\right]\varepsilon\cos\beta\sin\alpha - P_1(Q_1\cos L - Q_2\sin L)\varepsilon\sin\beta\right\}$$

$$\frac{dQ_1}{dt} = \frac{r}{2h}\left(1 + Q_1^2 + Q_2^2\right)\sin L \cdot \varepsilon\sin\beta$$

$$\frac{dQ_2}{dt} = \frac{r}{2h}\left(1 + Q_1^2 + Q_2^2\right)\cos L \cdot \varepsilon\sin\beta$$

$$\frac{dL}{dt} = \sqrt{\frac{\mu}{a^3}} - \frac{r}{h}(Q_1\cos L - Q_2\sin L)\varepsilon\sin\beta$$

Low Thrust Two-Points Boundary Value Problem

- Objective: $$\min_u \Delta V = \int_{t_0}^{t_f} \varepsilon(t)\, dt$$
- Where the control vector u is: $u = [\varepsilon(t), \alpha(t), \beta(t)]^T$
- Satisfying Gauss' Planetary Equations and the boundary constraints:

$$\begin{cases} \mathbf{X}(t_0) = \overline{\mathbf{X}}_0 \\ \mathbf{X}(t_f) = \overline{\mathbf{X}}_f \end{cases}$$

and with: $$\varepsilon(t) \le \varepsilon_{\max}$$

- Assumptions:
  - Perturbing acceleration ε is very small compared to the local gravitational acceleration:

$$\varepsilon << \frac{\mu}{r^2}$$

  - Constant modulus and direction in the radial-tangential reference frame.

$$[\varepsilon, \alpha, \beta] = const$$

From Time to True Longitude

- A system of differential equations in time is translated into a system of differential equations in true longitude:

$$\frac{d\mathbf{X}}{dt} = f(\mathbf{X}, L, \varepsilon, \alpha, \beta) \Rightarrow \frac{d\mathbf{X}}{dL} = f(\mathbf{X}, L, \varepsilon, \alpha, \beta)$$

- It is possible to rewrite Gauss Planetary Equations with respect to Longitude by applying the chain rule:

$$\frac{d}{dL} = \frac{d}{dt}\frac{dt}{dL}$$

- Where $\frac{dL}{dt}$, with the hypothesis of small, constant, perturbing acceleration:

$$\frac{dL}{dt} = \dot{\Omega} + \dot{\omega} + \dot{\vartheta} \approx \dot{\vartheta} = \frac{h}{r^2}$$

- Thus obtaining also the time equation:

$$\frac{dt}{dL} = \frac{r^2}{h} = \frac{h^3}{\left(1 + P_1 \sin L + P_2 \cos L\right)^2}$$

First order expansion of Equations of Motion (1)

- The aim is that of obtaining a first-order expansion of the variation of Equinoctial elements in the form with respect to a reference orbit:

$$\mathbf{X} = \mathbf{X_0} + \varepsilon \mathbf{X_1}$$

- With the manipulations described above, one obtains a set of equations in the form:

$$\mathbf{X}' = \mathbf{X_0}' + \varepsilon \mathbf{X_1}'$$

- Which could be integrated between L_0 and L.

Excursus on Perturbation Theory

- Consider the following example:

$$\ddot{x} = -x + \varepsilon x^2$$

- And the first order expanded solution:

$$x = x_0 + \varepsilon x_1$$
$$\dot{x} = \dot{x}_0 + \varepsilon \dot{x}_1$$
$$\ddot{x} = \ddot{x}_0 + \varepsilon \ddot{x}_1$$

If this expanded solution is substituted in the original equation one gets:

$$\ddot{x}_0 + \varepsilon \ddot{x}_1 = -x_0 - \varepsilon x_1 + \varepsilon (x_0 + \varepsilon x_1)^2$$
$$\ddot{x}_0 + \varepsilon \ddot{x}_1 = -x_0 - \varepsilon x_1 + \varepsilon x_0^2 + 2\varepsilon^2 x_0 x_1 + \varepsilon^2 x_1^2$$

Excursus on Perturbation Theory

- Now assuming we are interested only in first order terms we can collect terms with similar order of ε:

$$\ddot{x}_0 = -x_0$$
$$\ddot{x}_1 = -x_1 + x_0^2$$

The first equation is a simple harmonic oscillator and once its solution is substituted in the second equation one has a simple harmonic oscillator with periodic forcing term.

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First order expansion of Equations of Motion (2)

- A first order expansion could be obtained by simply integrating the ODEs with respect to *L*, which could be done in closed form.
- This requires finding the primitives of the integrals in the form:

$$I_1\left(L_F\right)=\int_{L_0}^{L_F}\frac{1}{\left(1+P_{10}\sin L+P_{20}\cos L\right)^3}dL$$

$$I_C\left(L_F\right)=\int_{L_0}^{L_F}\frac{\cos L}{\left(1+P_{10}\sin L+P_{20}\cos L\right)^3}dL$$

$$I_S\left(L_F\right)=\int_{L_0}^{L_F}\frac{\sin L}{\left(1+P_{10}\sin L+P_{20}\cos L\right)^3}dL$$

- For example:

$$I_S(L)=\int_0^L\frac{\sin L}{\left(1+P_{10}\sin L+P_{20}\cos L\right)^2}dL=$$

$$=\frac{P_{10}\ln\left[\left(\sqrt{P_{10}^2+P_{20}^2-1}-P_{10}\right)\cos L+(P_{20}-1)\sin L+\sqrt{P_{10}^2+P_{20}^2-1}-P_{10}\right]}{\left(P_{10}^2+P_{20}^2-1\right)^{3/2}}+$$

$$-\frac{P_{10}\ln\left[\left(\sqrt{P_{10}^2+P_{20}^2-1}-P_{10}\right)\cos L+(1-P_{20})\sin L+\sqrt{P_{10}^2+P_{20}^2-1}+P_{10}\right]}{\left(P_{10}^2+P_{20}^2-1\right)^{3/2}}+$$

$$+\frac{(P_{20}+1)\cos L+P_{10}\sin L+P_{20}+1}{\left(P_{10}^2+P_{20}^2-1\right)\left(P_{20}\cos L+P_{10}\sin L+1\right)}$$

Analytical Solution of the Equations of Motion

- Thus the first order approximate solution of perturbed Keplerian motion takes the form:

$$a(L) = a_0 + \varepsilon a_1 = a_0 + \varepsilon \left\{ 2h_0^2 a_0^2 \cos\beta \cos\alpha \left[P_{20} I_{s2}(L_0, L) - P_{10} I_{c2}(L_0, L) \right] + 22 h_0^2 a_0^2 \cos\beta \sin\alpha I_{11}(L_0, L) \right\}$$

$$P_1(L) = P_{10} + \varepsilon P_{11}$$

$$P_2(L) = P_{20} + \varepsilon P_{21}$$

$$Q_1(L) = Q_{10} + \varepsilon Q_{11}$$

$$Q_2(L) = Q_{20} + \varepsilon Q_{21}$$

$$t(L) = t_0 + \varepsilon t_1$$

- A complete set of analytic equations parameterised on the Longitude is thus available to propagate the perturbed orbital motion, in the form:

$$\mathbf{X}(L_0 + \Delta L) = f\left(\mathbf{X}(L_0), \Delta L, \varepsilon, \alpha, \beta\right)$$

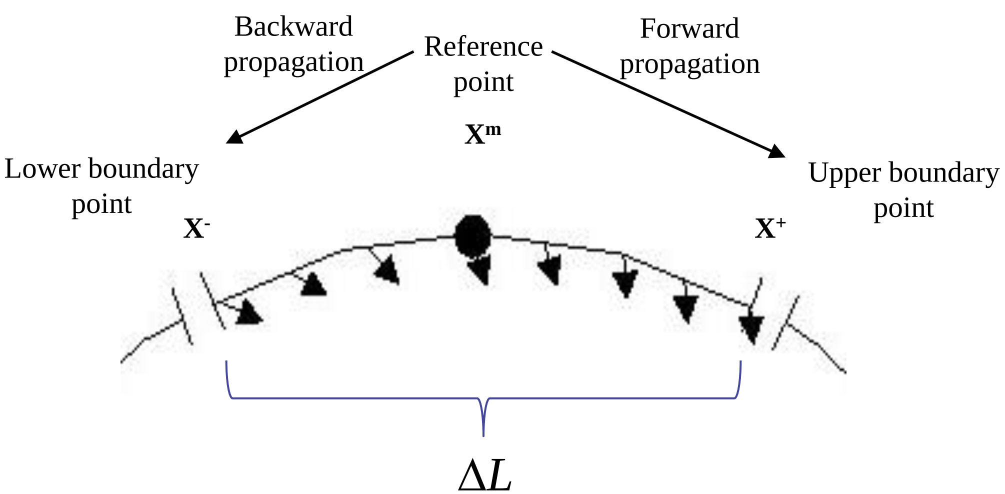
Some remarks on accuracy

- The accuracy of the approximation is dependent on:
    - The ratio between the local gravitational acceleration and the perturbation. Thus:
        - Error increases according to the thrust level → works better with small thrust magnitude (<1 N).
        - Error increases with the distance from the central body → works better very close to the attractor.
    - The amplitude of the trajectory arc. Error increases superlinearly with the amplitude of the propagation interval.

- Tests have revealed that in most practical applications thrust and local gravitational force still present a favourable ratio.
- The error in amplitude is mitigated by dividing a single arc into sub-arcs of smaller amplitude.

FPET method

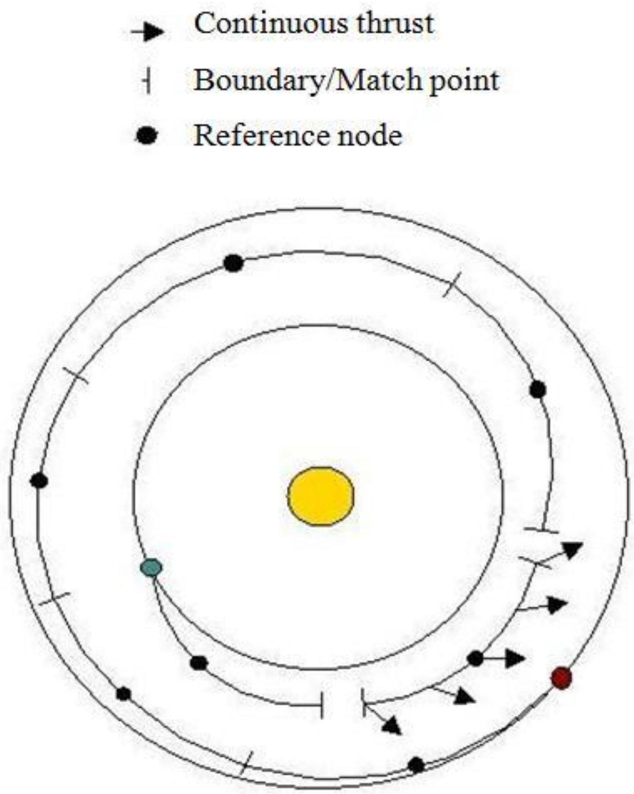
- A Direct Transcription Method based on Finite Perturbed Elements in Time (FPET) has been designed using the Perturbative approach.
- Each transfer trajectory is divided into n subarcs:
  - Amplitude of arc is *ΔL.*
  - Perturbed motion propagated using analytical solution.
  - Constant thrust vector in the r-t-h reference frame.
  - Reference node for propagation is the midpoint of the arc.
  - Motion is propagated analytically backwards and forwards by $\pm\frac{\Delta L}{2}$ from the midpoint to obtain boundary nodes. (better accuracy compared to a single sided propagation)



$$\begin{cases} \mathbf{X}^{+} = f\left(\mathbf{X}^{\mathbf{m}}, \frac{\Delta L}{2}, \varepsilon, \alpha, \beta\right) \\ \mathbf{X}^{-} = f\left(\mathbf{X}^{\mathbf{m}}, -\frac{\Delta L}{2}, \varepsilon, \alpha, \beta\right) \end{cases}$$

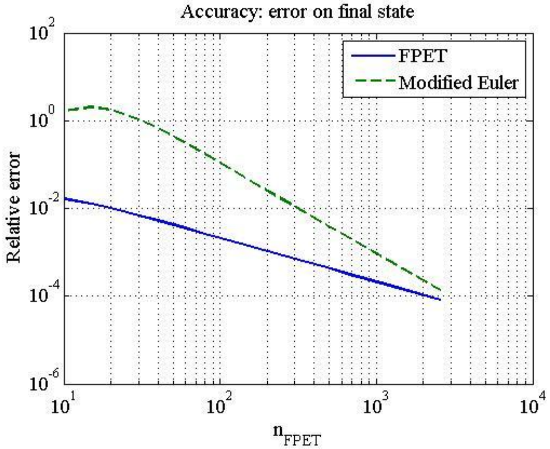
FPET method

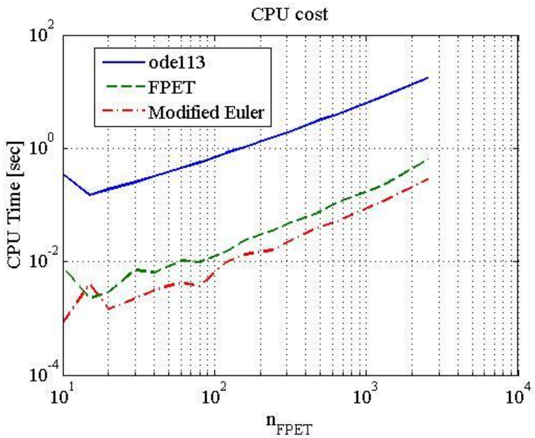
- The subarcs are then matched to each other at the boundaries to obtain the complete trajectory.
- Conceptually similar to Sims & Flanagan Direct Transcription Method which used Keplerian arcs with ΔV discontinuities at the boundaries.
- In the FPET method, thrust is continuous, albeit constant within each element.
- In both cases, orbit propagation is analytical.



Accuracy and CPU time (1)

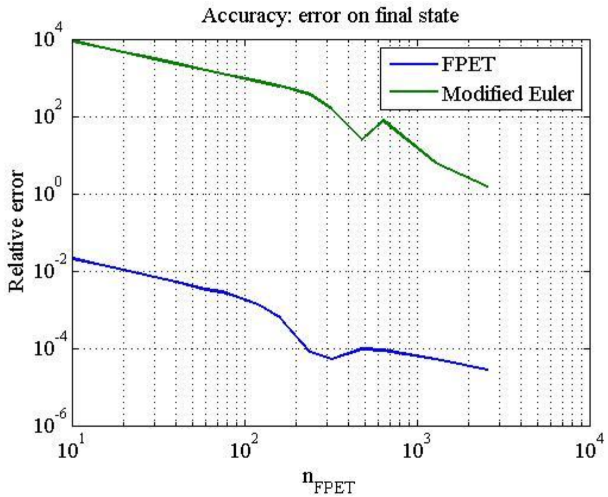
- To assess accuracy and CPU cost, orbital motion under constant thrust was propagated within a given time interval.
- A range of values for the number sub-arcs was considered.
- Accuracy was calculated in terms of relative error on final state with respect to numerical integration of the exact equations of motion.
- CASE 1, Heliocentric orbital motion:
  - 0.5 N continuous thrust on a 2000 kg spacecraft. $\varepsilon=2.5\ 10^{-5}\ m/s^2$
  - Departure from Earth.
  - 1.5 years.

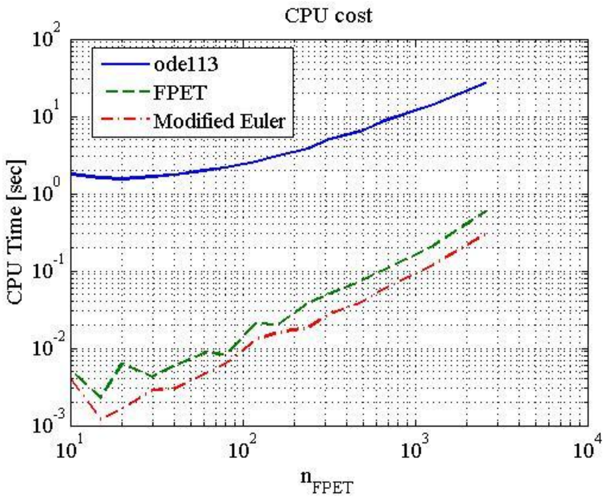




The LT Boundary problem with FPET

- CASE 2, Geocentric orbital motion:
- 0.5 N continuous thrust on a 2000 kg spacecraft. $\varepsilon=2.5\ 10^{-5}$ m/s^2
- Departure from LEO
- 5 days (≈50 revolutions).





- FPET method is 2 orders of magnitude faster than numerical integration.
- Even with few elements per revolution, accuracy is high for geocentric orbits.
- In heliocentric orbits, accuracy is still adequate.
- The FPET method is particularly advantageous with a relatively low number of sub-arcs.

The Low Thrust Two-Points Boundary Value Problem with FPET

- The FPET transcription method is used to solve the LT boundary problem:

$$\min_{\mathbf{u}} J = \sum_{i=1}^{n_{FPET}} \varepsilon_i \Delta t_i$$

$$s.t.\ \mathbf{C}_{\mathbf{eq}} = \left\{ \begin{array}{c} \mathbf{X}_1^- - \mathbf{X}_0 \\ \mathbf{X}_i^+ - \mathbf{X}_{i+1}^-, i = 2, \ldots, n_{FPET} - 1 \\ \mathbf{X}_{n_{FPET}}^+ - \mathbf{X}_f \\ \overline{ToF} - \sum_{i=1}^{n_{FPET}} \Delta t_i \end{array} \right\} = 0$$

$$\varepsilon_i \le \varepsilon_{max},\ i = 1, \ldots, n_{FPET}$$

Total ΔV

Departure conditions

Inter-element matching conditions

Arrival conditions

Fixed time of flight

Upper bounds on acceleration

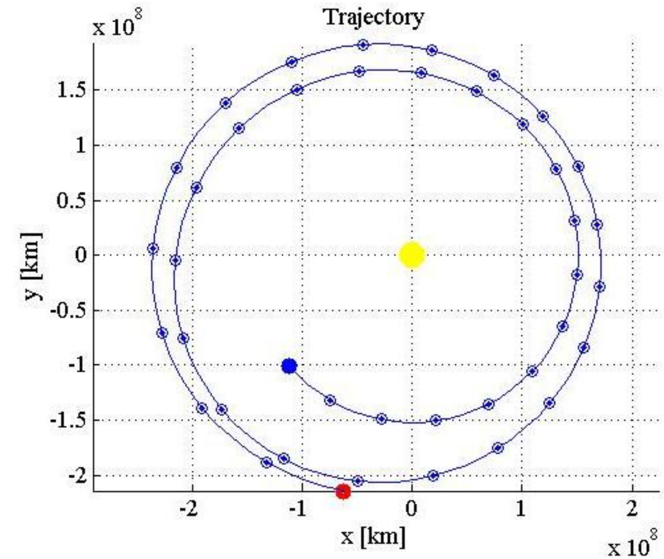
- Decision variables for each of the n FPET:
  - Position of the reference point (5 scalars).
  - Acceleration magnitude, azimuth and elevation (3 scalars).
- 8n decision variables and 5(n+1)+1 scalar constraints.
- The problem is efficiently solve with a gradient-based local optimizer (*fmincon active-set*).

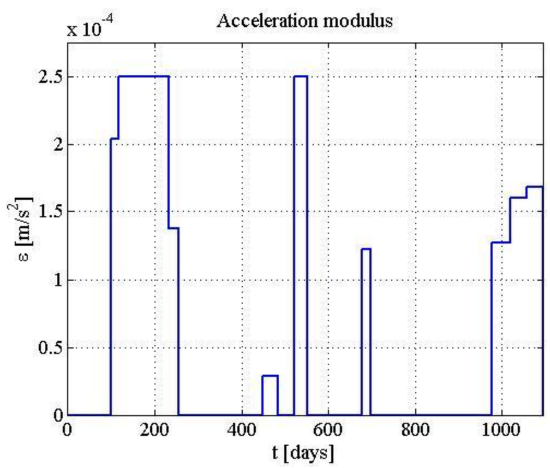
Earth-Mars Direct transfer (1)

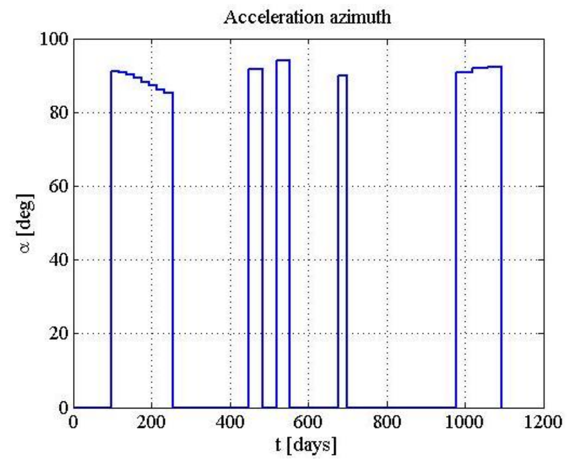
- Boundary problem:
  - Departure from Earth at 5600 MJD2000.
  - Rendezvous with Mars after a transfer time of 3 years.
  - 2 complete revolutions.
- Maximum acceleration: 2.5 10^{-5} m/s^2.
- 40 FPET.
- Initial guess for the local optimizer: constant thrust profile.

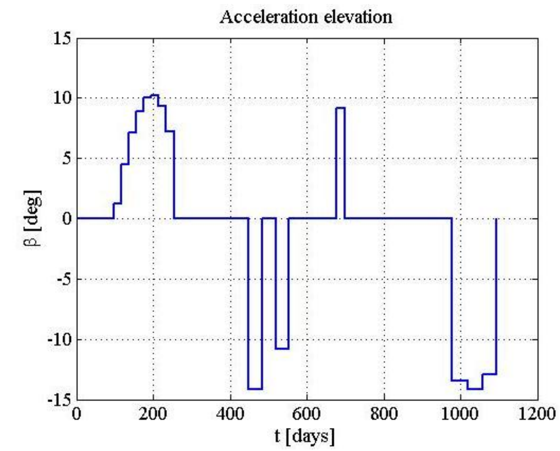
Earth-Mars Direct transfer (2)

- Results:
  - Total ΔV: 5.63 km/s.
  - Relative error 10^{-3}.
  - Solution found with DITAN: 5.71 km/s.
  - Hohmann Transfer: 5.49 km/s.









Earth-Mars Direct transfer: Multi-Objective Problem (1)

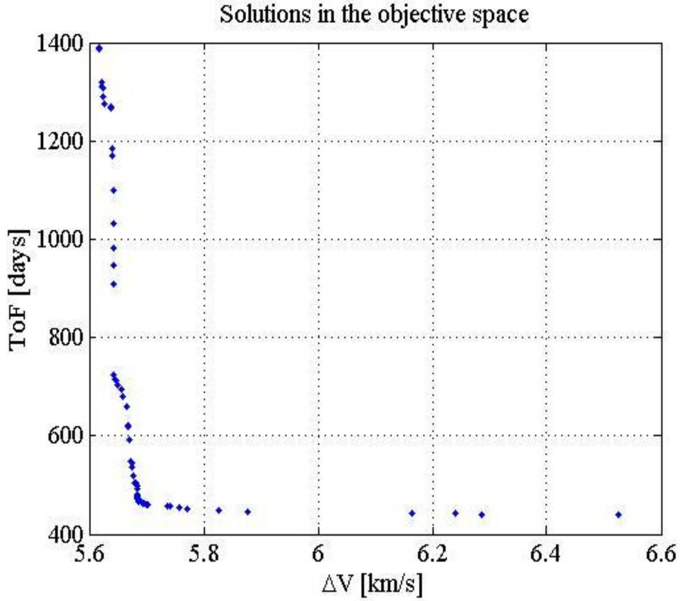
- Bi-objective optimization problem: ΔV and Time of Flight are minimised.

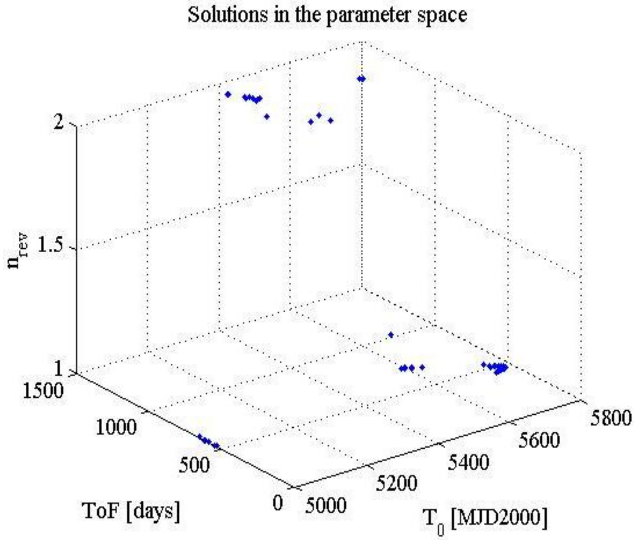
	Lower	Upper
T_0 [MJD2000]	5000	5779.94
ToF [days]	100	1500
n_{rev}	1	3

- Each time to objective function is called, a boundary problem needs to be solved.
- Solved with EPIC, a hybrid-memethic stochastic optimizer.
- 8000 function evaluations.

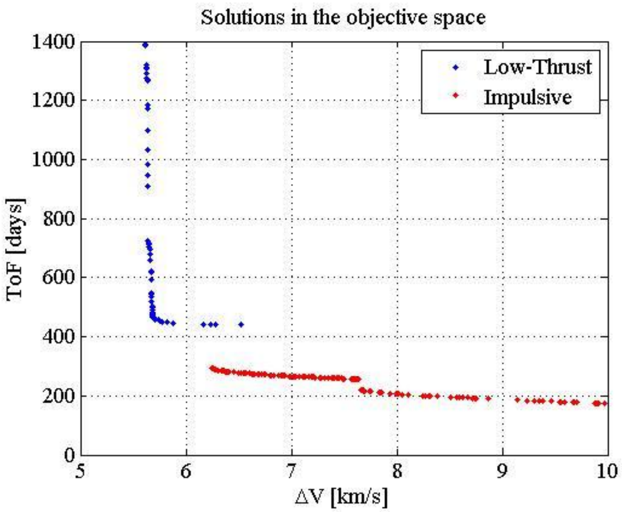
Earth-Mars Direct transfer: Multi-Objective Problem (2)

- Bi-objective optimization problem: ΔV and Time of Flight are minimised.





- An interesting comparison could be made with an analogous Biimpulsive Transfer to Mars.



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